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The second unit in the SMSG series for junior high school mathematics is a student text covering the following topics: ratios, percents, and decimals; parallels, parallelograms, triangles, and right prisms; circles; mathematical systems; statistics and graphs; and mathematics in science. (DT)

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STUDENT'S TEXT UNIT NO. 2

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MATHEMATICS FOR JUNIOR HIGH SCHOOL
VOLUME 1

PART II

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SCHOOL MATHEMATICS STUDY GROUP

YALE UNIVERSITY PRESS

School Mathematics Study Group

Mathematics for Junior High School, Volume 1

Unit 2

Mathematics for Junior High School, Volume 1

Student's Text, Part II

Prepared under the supervision of the
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Chapter 9

RATIOS, PERCENTS, AND DECIMALS

9-1. Ratios

One sunny day a boy measured the length of the shadow cast by each member of his family. He also measured the length of the shadow cast by a big tree in their yard. He found that his father, who is 72 inches tall, cast a shadow 48 inches long. His mother, who is 63 inches tall, cast a shadow 42 inches long. His little brother, who is only 30 inches high, cast a shadow 20 inches long. He didn't know how tall the tree was, but its shadow was 40 feet long.

Let us arrange this information in a table.

	Shadow length	Height
Father	48 inches	72 inches
Mother	42 inches	63 inches
Brother	20 inches	30 inches
Tree	40 feet	?

We see that the taller people have longer shadows. But let us examine this more closely. Suppose we divide the number of inches in the shadow length of the little brother by the number of inches in his height. We get $\frac{20}{30}$ or $\frac{2}{3}$. Suppose we try the same thing for the father. It will be easier to measure the father's height and shadow in feet. The father is 6 feet tall and his shadow is 4 feet long. If we divide the measure of the shadow length in feet by the measure of his height in feet, we get $\frac{4}{6}$ or $\frac{2}{3}$.

Divide the number of inches in the shadow length of the mother by the number of inches in her height. Do you again get $\frac{2}{3}$?

Let us assume that this principle holds for all objects (we must measure the shadows at the same time and place; the shadow changes during the day as the position of the sun changes).

Then the tree must be 60 feet tall in order that the measure of its shadow length divided by the measure of its height be $\frac{2}{3}$. Thus we can discover how tall the tree is without actually measuring it!

In Chapter 6 we used the term "ratio" and studied some applications of ratio.

Definition. The ratio of a number a to a number b $(b \neq 0)$, is the quotient $\frac{a}{b}$. (Sometimes this ratio is written $a:b$.)

We have formed the ratio of the measure of shadow length to the measure of height, and we discovered that this ratio was the same for all the people whom we measured. Using this we were able to discover that the tree was 60 feet tall.

Suppose that the boy's uncle is 66 inches tall (5 feet 6 inches). How long would his shadow be if it were measured at the same time and place as the other people?

To answer this question, we let s be the number of inches in his shadow length. Then we must have:

$$\frac{s}{66} = \frac{2}{3}.$$

The number $\frac{2}{3}$ can be expressed as a fraction with denominator 66.

$$\frac{2}{3} = \frac{2}{3} \cdot \frac{22}{22} = \frac{44}{66}$$

$$\frac{s}{66} = \frac{44}{66}$$

$$s = 44$$

The uncle's shadow would be 44 inches long.

You have seen that a ratio is a comparison of two numbers by division. The numbers may be measures of physical quantities. The word, "per," indicates division. We use it to express the ratio of the measures of two physical quantities, such as miles and hours. Store prices provide additional examples. Prices relate value to amount such as $1.00 per pound, or $.59 per dozen. In each case, the per indicates a ratio, generally between two different kinds of quantities. This is sometimes called a rate. Notice further that the second quantity in each case represents the standard of comparison. A store charges "10¢ per comb"; one comb is the standard of comparison, and they want 10¢ in the cash register for each comb sold. Of course, this standard does not always represent a quantity of one. For example, cents per dozen, dollars per pair (for shoes), dollars per 1000 (for bricks).

Taking another look at our example and at the definition of ratio, we see that a ratio compares just two numbers. In our example we had two sets of numbers, one set from the lengths of the shadows, and one set from the heights. In each case we formed the ratio of the first number (shadow length) to the second number (height). In our example these ratios were all equal to one another. In such a situation, we say that the physical quantities measured by the numbers are proportional to one another. A proportion is a statement of equality of two ratios.

Thus in our example on the first page of this chapter, the shadow length is proportional to the height. The ratio of each corresponding pair of measures, in that example, is $\frac{2}{3}$. We will use this ratio again to find the height of a water tower if the shadow length of a water tower is 21 feet. Let w be the number of feet in the height of the tower.
Then the statement

$$\frac{21}{w} = \frac{2}{3}$$

is a proportion. Let us now consider statements of this form, and then a little later we will return to this statement and show a method for finding a number to substitute for w which will make the statement true.

We know from Chapter 6 that

(a) $\frac{2}{3} = \frac{6}{9}$; (b) $\frac{3}{4} = \frac{24}{32}$; (c) $\frac{5}{6} = \frac{15}{18}$; (d) $\frac{3}{8} = \frac{375}{1000}$

In (a) $\frac{2}{3} = \frac{6}{9}$, we see that $2 \cdot 9 = 18$ and $3 \cdot 6 = 18$; thus $2 \cdot 9 = 3 \cdot 6$. In (b) $\frac{3}{4} = \frac{24}{32}$, we see that $3 \cdot 32 = 96$ and $4 \cdot 24 = 96$; thus $3 \cdot 32 = 4 \cdot 24$. Similarly, in (d) $\frac{3}{8} = \frac{375}{1000}$, we see that $3 \cdot 1000 = 8 \cdot 375$, since $8 \cdot 375 = 3000$. Check this property for (c) $\frac{5}{6} = \frac{15}{18}$.

Try other examples of pairs of fractions which name the same rational number. Are products of the numerator of one fraction and the denominator of the other equal? Let us see if this property is true for all pairs of fractions, which are numerals for the same rational number.

If $\frac{a}{b} = \frac{c}{d}$ and $b \neq 0$ and $d \neq 0$, then

$$\frac{a}{b} \cdot \frac{d}{d} = \frac{c}{d} \cdot \frac{b}{b}$$

$\frac{a}{b}$ is multiplied by $\frac{d}{d}$, which equals 1, and $\frac{c}{d}$ is multiplied by $\frac{b}{b}$.

$$\frac{ad}{bd} = \frac{cb}{db}$$

$$\frac{ad}{bd} = \frac{bc}{bd}$$

These products are obtained so that the two rational numbers $\frac{a}{b}$ and $\frac{c}{d}$ will be expressed as fractions with a common denominator.

Thus $ad = bc$.

If two fractions with the same denominator are names for the same rational number, their numerators are equal.

We have shown:

Property 1.

If $\frac{a}{b} = \frac{c}{d}$ and $b \neq 0$ and $d \neq 0$, then $ad = bc$.

This property may be useful to you in solving problems involving proportion. It is also true that if $ad = bc$, then $\frac{a}{b} = \frac{c}{d}$ provided $b \neq 0$ and $d \neq 0$. We shall leave the proof of this property as a problem. (See Problem *18.)

We can now return to the problem of the height of the water tower.

If $\frac{21}{w} = \frac{2}{3}$

then $21 \cdot 3 = 2w$ Property 1.

$63 = 2w$

$\frac{63}{2} = w$ Definition of a rational number.

$31\frac{1}{2} = w$

The water tower is $31\frac{1}{2}$ feet in height.

Exercises 9-1

1. What is your height in inches? What would be the length of your shadow if it were measured at the same time and place as the shadows of the people in our story were measured?
2. Some other objects were measured at another time and place, and the data are recorded below. Copy and complete the table.

	Shadow length	Height	Ratio
Garage	3 feet	8 feet	
Clothes pole	36 inches		$\frac{3}{8}$
Tree	$7\frac{1}{2}$ feet	20 feet	
Flag pole		144 inches	$\frac{3}{8}$
Fence	$11\frac{1}{4}$ inches	30 inches	

3. In a class there are 30 students of whom 12 are girls.
   (a) What is the ratio of the number of girls to the total number of students in the class?
   (b) What is the ratio of the number of boys to the total number of students in the class?
   (c) What is the ratio of the number of girls to the number of boys?
4. In another class, the ratio of the number of girls to the number of boys is $\frac{2}{3}$, the same as the ratio for the class in Problem 3. This class has 10 girls. How many boys does it have?
5. In Problem *18 you are asked to show that $\frac{a}{b} = \frac{c}{d}$, if $ad = bc$ and $b \neq 0$ and $d \neq 0$. Use this property to find which of the following pairs of ratios are equal.

(a) $\frac{10}{5}, \frac{20}{10}$

(b) $\frac{6}{3}, \frac{16}{9}$

(c) $\frac{48}{16}, \frac{42}{14}$

(d) $\frac{68}{17}, \frac{76}{19}$

(e) $\frac{15}{13}, \frac{45}{39}$

6. In each of the following, find x so that these will be true statements.

(a) $\frac{20}{8} = \frac{x}{6}$

(b) $\frac{14}{30} = \frac{x}{90}$

(c) $\frac{x}{3} = \frac{75}{15}$

(d) $\frac{81}{108} = \frac{x}{12}$

(e) $\frac{x}{42} = \frac{36}{27}$

7. Joyce has a picture 4 inches wide and 5 inches long. She wants an enlargement that will be 8 inches wide. How long will the enlarged print be?

8. If a water tower casts a shadow 75 feet long and a 6-foot man casts a shadow 4 feet long, how tall is the water tower?

9. Mr. Landry was paid \$135 for a job which required 40 hours of work. At this rate, how much would he be paid for a job that required 60 hours of work?

10. A cookie recipe calls for the following items.

1 cup butter

$\frac{2}{3}$ cup sugar

2 eggs

$1\frac{1}{2}$ cups flour

1 teaspoon vanilla

This recipe will make 30 cookies.

(a) Rewrite the recipe by enlarging it in the ratio $\frac{3}{1}$.

(b) How many cookies will the new recipe make now?

(c) Suppose you wanted to make 45 cookies, how much would you need of each of the items listed above?

11. Use a proportion to solve these problems. Check your work by solving the problems by another method.

(a) What is the cost of 3 dozen doughnuts at \$.55 per dozen?

(b) What is the cost of 12 candy bars at 4 for 15¢? (How could you state this price using the word "per"?)

(c) What is the cost of 8500 bricks at $14 per thousand?

(d) A road has a grade of 6 %, which means that it rises 6 feet per 100 feet of road. How much does it rise in a mile? Find the answer to the nearest foot.

12. Common units to express speed are "miles per hour" and "feet per second." A motor scooter can go 30 miles per hour. How many feet per second is this? (The danger of of high speed under various conditions often can be realized better when the speed is given in feet per second.)

13. The following table lists pairs of numbers, A and B. In each pair the ratio of A to B is the number $\frac{6}{7}$. Complete the table.

	A	B	Ratio $\frac{A}{B}$	Ratio in simplest form
(a)	12	14	$\frac{12}{14}$	$\frac{6}{7}$
(b)		21		
(c)	30			
(d)		100		
(e)	100			

*14. Sometimes it is natural and desirable to compare more than just two quantities. For example, a mixture of nuts calls for 5 pounds of peanuts, 2 pounds of cashews, and 1 pound of pecans. Here the ratio of the number of pounds of peanuts to the number of pounds of cashews is $\frac{5}{2}$ or 5 to 2, and the ratio of the number of pounds of cashews to the number of pounds of pecans is 2 to 1. This may be stated briefly as 5 to 2 to 1. Suppose a grocer wants to prepare 24 pounds of this mixture. How many pounds of peanuts, cashews and pecans should he use? To determine your answer, first answer these questions:

(1) If 5 pounds of peanuts, 2 pounds of cashews, and 1 pound of pecans are mixed together, how many pounds of nuts are there?

(2) What is the ratio of the number of pounds of peanuts to the total number of pounds?

(3) Since this ratio will be the same in the mixture whose total weight is 24 pounds, how many pounds of peanuts are required?

(4) Answer questions (2) and (3) when "peanuts" are replaced by "cashews".

(5) Answer questions (2) and (3) when "peanuts" are replaced by "pecans."

*15. Fifty-six pounds of a nut mixture, with the same ingredients in the same ratio as in Problem 14 will be prepared. How many pounds of each kind of nuts will the grocer need?

*16. If a nut mixture with the same ingredients as in Problem 14, but in the ratio 5 to 3 to 2 has a total weight of 100 pounds, how many pounds of each kind of nuts are in the mixture?

*17. A triangle has sides of length 11 inches, 8 inches, and 6 inches. If another triangle, the measures of whose sides are in the ratio 11 to 8 to 6, is to be drawn with a perimeter of 100 inches, how long will the shortest side be?

*18. Prove the property,

if $ad = bc$ and $b \neq 0$ and $d \neq 0$, then $\frac{a}{b} = \frac{c}{d}$.

When you have proved this statement, you can now say

if $\frac{a}{b} = \frac{c}{d}$, and $b \neq 0$ and $d \neq 0$, then $ad = bc$,

and only then.

(Hint: If $ad = bc$, then $a = \frac{bc}{d}$. We can express $\frac{bc}{d}$ as $b \cdot \frac{c}{d}$.)

9-2. Percent

Many of you are familiar with the word "percent," and you may know something about its meaning. If your teacher says, "90 percent of the answers on this paper are correct," would you know what he means? The word "percent" comes from the latin phrase "per centum," which means "by the hundred." If the paper with 90 of the answers correct has 100 answers, then 90 answers out of the 100 are correct. The ratio $\frac{90}{100}$ could be used instead of the phrase "90 percent" to describe the part of the answers which are correct. The word "percent" is used when a ratio is expressed with a denominator of 100.

$$90 \text{ percent} = \frac{90}{100} = 90 \cdot \frac{1}{100} .$$

For convenience the symbol, %, is used for the word "percent." This symbol is a short way of saying $\frac{1}{100}$.

$$\frac{90}{100} = 90 \cdot \frac{1}{100} = 90\% .$$

$$\frac{16}{100} = 16 \cdot \frac{1}{100} = 16\% .$$

$$\frac{37}{100} = 37 \cdot \frac{1}{100} = 37\% .$$

$$\frac{77}{100} = 7 \cdot \frac{1}{100} = ?\% .$$

$$? = 13 \cdot \frac{1}{100} = 13\% .$$

Suppose that the paper has 90 correct answers out of the 100; 6 incorrect answers out of the 100; 4 answers omitted out of the 100. Since we know the total number of problems we can express this information in terms of percent.

$\frac{90}{100} = 90 \cdot \frac{1}{100} = 90\%$ (90% of the problems were correct.)

$\frac{6}{100} = 6 \cdot \frac{1}{100} = 6\%$ (6% of the problems were incorrect.)

$\frac{4}{100} = 4 \cdot \frac{1}{100} = 4\%$ (4% of the problems were omitted.)

$\frac{90}{100} + \frac{6}{100} + \frac{4}{100} = \frac{100}{100}$ (all possible answers.)

$90\% + 6\% + 4\% = 100\%$ (all possible answers.)

Another name for the number one is 100%.

The number 2 can be written

$$\frac{2}{1} = \frac{200}{100} = 200\% .$$

In other words 200% means $200 \times \frac{1}{100} = \frac{200}{100} = 2$.

A class of 25 pupils is made up of 11 girls and 14 boys. The ratio of the number of girls to the number of pupils in the class can be expressed many ways. For instance:

$$\frac{11}{25} = \frac{22}{50} = \frac{33}{75} = \frac{44}{100} = \frac{55}{125} = \frac{66}{150}.$$

If we wish to indicate the percent of the class that is girls, which fraction gives the information most easily? Why? The ratio of the number of boys to the total number in the class may be written

$$\frac{14}{25} = \frac{c}{50} = \frac{d}{75} = \frac{56}{100} = \frac{e}{125} = \frac{f}{150}.$$

What numbers are represented by the letters c, d, e, f? Notice the two ratios $\frac{11}{25}$ (girls) and $\frac{14}{25}$ (boys). What is the sum of the two ratios? Find the sum of the two ratios $\frac{44}{100}$ and $\frac{56}{100}$. Express the two ratios and their sum as percents using the symbol, %. The entire class is considered to be 100% of the class because

$\frac{25}{25} = \frac{100}{100} = 100\%$.

Any number $\frac{a}{b}$ can be expressed as a percent by finding the number c such that

$$\frac{a}{b} = \frac{c}{100} = c \cdot \frac{1}{100} = c\%.$$

Exercises 9-2a

1. Using squared paper draw a large square whose interior is divided into 100 small squares. Write the letter A in 10 small squares. Write the letter B in 20% of the squares. Write the letter C in 35% of the squares. Write the letter D in 30 of the squares. Write the letter X in the remainder of the squares.

   (a) In what fractional part of the squares is the letter A?

(b) In what percent of the squares is the letter A?
(c) In how many squares is the letter B?
(d) In what fractional part of the squares is the letter B?
(e) In how many squares is the letter C?
(f) In what fractional part of the squares is the letter C?
(g) In what fractional part of the squares is the letter D?
(h) In what percent of the squares is the letter D?
(i) In what fractional part of the squares is the letter X?
(j) In what percent of the squares is the letter X?

2. What is the sum of the numbers in your answers for (a), (d), (f), (g), (i), of Problem 1?

3. In Problem 1, what is the sum of the percents of the squares containing the letters A, B, C, D, X?

4. Write each of the following numbers as a percent.

(a) $\frac{1}{2}$ (d) $\frac{1}{5}$ (g) $\frac{3}{2}$ (j) $\frac{7}{5}$
(b) $\frac{1}{4}$ (e) $\frac{2}{5}$ (h) $\frac{2}{2}$ (k) $\frac{5}{4}$
(c) $\frac{3}{4}$ (f) $\frac{3}{5}$ (i) $\frac{4}{4}$ (l) $\frac{5}{2}$

5. Consider the following class.

Student	Hair Color
Robert	blond
Joan	brown
Mary	blond
John	red
Joe	brown
Betty	brown
Ray	blond
Don	brown
Margaret	red
David	brown

(a) What percent are boys?
(b) What percent are blonds?
(c) What percent are redheads?
(d) What percent are not redheads?
(e) What percent are brown-haired girls?
(f) What percent are redheaded boys?

6. Jean has a weekly allowance of 50 cents. One week she spends 12 cents for a pencil, 10 cents for an ice cream cone, 15 cents for Sunday-school collection, and she puts the rest in her bank.
   - (a) Write the ratio of each expense and the money banked to the allowance, and express each ratio as a percent.
   - (b) Find the sum of the ratios.
   - (c) Find the sum of the numbers named by the percents.
   - (d) What check on the separate answers above do you observe?
7. The monthly income for a family is $400. The family budget for the month is shown.

Payment on the mortgage for the home	$ 80
Taxes	20
Payment on the car	36
Food	120
Clothing	48
Operating expenses	32
Health, Recreation, etc.	24
Savings, and Insurance	40

   - (a) What percent of the income is assigned to each item of the budget?
   - (b) What is one check on the accuracy of the 8 answers?

Percent is a convenient tool for giving information involving ratios. Athletic standings sometimes are given in percent. Two seventh-grade pupils discovered the reason for this use of percent. The boys were discussing the scores of their baseball teams. In the Little League one team won 15 games out of 20 games played. Another team won 18 out of 25 games. Which team had a better record? The second team had won 3 more games, but the first team played fewer games. Look at the ratios of the number of games won to the number of games played for each team. The ratios $\frac{15}{20}$ and $\frac{18}{25}$ cannot be compared at a glance. Let us use percent for the comparison.

The first team won $\frac{15}{20}$ of the games played.

$\frac{15}{20} = \frac{75}{100} = 75\%$. They won 75% of the games played.

The second team won $\frac{18}{25}$ of the games played.

$\frac{18}{25} = \frac{72}{100} = 72\%$ They won 72% of the games played.

The first team which won 75% of its games had a higher standing than the second team which won 72% of its games. We could say that $72\% < 75\%$ or $75\% > 72\%$.

Data about business, school, Scouts -- are sometimes given in percent. It is often more convenient to refer to the data at some later time if they are given in percent than if they are given otherwise.

A few years ago the director of a camp kept some records for future use. Some information was given in percent, and some was not. The records gave the following items of information.

(1) There were 200 boys in camp.

(2) One hundred percent of the boys were hungry for the first dinner in camp.

(3) On the second day in camp 44 boys caught fish.

(4) One boy wanted to go home the first night.

(5) A neighboring camp director said "forty percent of the boys in my camp will learn to swim this summer. We shall teach 32 boys to swim."

From (1) and (2), how many hungry boys came to dinner the first day?

$$\frac{100}{100} \cdot 200 = 200$$

Of course we should know without computation that 100% of 200 is 200.

From (3), we can find the percent of boys who caught fish on the second day. The ratio of the number of boys who caught fish to the number of boys in camp is $\frac{44}{200}$. If we call $x\%$ the percent of the boys who caught fish, then

$$\frac{x}{100} = \frac{44}{200}$$

$$200x = 100 \cdot 44 \qquad \text{(Property 1.)}$$

$$x = \frac{100 \cdot 44}{200}$$

$$x = 22$$

Twenty-two % of the boys caught fish.

From (4), we can find the percent of the boys who were homesick. If this is $x\%$, then

$$\frac{x}{100} = \frac{1}{200}$$

$$200x = 100 \qquad \text{(Property 1.)}$$

$$x = \frac{100}{200}$$

$$x = \frac{1}{2}$$

Of the boys, $\frac{1}{2}\%$ were homesick. This may be read "one-half percent" or "one half of one percent." You may prefer to say "one half of one percent," since this emphasizes the smallness of it.

From the information in Item 5, the total number of boys in the second camp can be found. Call this number of boys x.

40% means $\frac{40}{100}$ of the group, and also refers to 32 boys. We wish to find x such that $\frac{40}{100} = \frac{32}{x}$.

$$\frac{40}{100} = \frac{32}{x}$$

$$40x = 3200 \qquad \text{(Property 1.)}$$

$$x = \frac{3200}{40}$$

$$x = 80 .$$

Exercises 9-2b

1. A Little League team won 3 out of the first 5 games played.
   (a) What percent of the first 5 games did the team win?
   (b) What percent of the first 5 games did the team lose?

2. Later in the season the team had won 8 out of the 16 games played.
   (a) What was the percent of games won at this time?
   (b) Has the percent of games won increased or decreased?
3. At the end of the season, the team had won 26 games out of 40.
   (a) What percent of the games played did the team win by the end of the season?
   (b) How does this percent compare with the other two?
4. There are 600 seventh-grade pupils in a junior high school. The principal hopes to divide the pupils into 20 sections of equal size.
   (a) How many pupils would be in each section?
   (b) What percent of the pupils would be in each section?
   (c) What number of pupils is 1% of the number of pupils in the seventh grade?
   (d) What number of pupils is 10% of the number of pupils in the seventh grade?
5. Suppose one section contains 36 pupils. What percent of the seventh-grade pupils are in that section?
6. One hundred fifty seventh-grade pupils come to school on the school bus.
   (a) What percent of the seventh-grade pupils come by school bus?
   (b) What percent of the seventh-grade pupils come to school by some other means?
7. In a section of 30 pupils, 3 were tardy one day.
   (a) What fractional part of that section was tardy?
   (b) What percent of that section was tardy?
8. There were 750 eighth-grade pupils. The number of eighth-grade pupils is what percent of the number of seventh-grade pupils?
9. One day 3 seventh-grade pupils came to school on crutches (they had been skiing). What percent of the number of seventh-grade pupils were on crutches?

10. One day a seventh-grade pupil heard the principal say, "Four percent of the ninth graders are absent today." A list of absentees for that day had 22 names of ninth-grade pupils on it. From these two pieces of information, the seventh-grade pupil discovered the number of ninth-grade pupils in the school. How many ninth-grade pupils are there?

9-3. Decimal Notation

You recall from Chapter 2 that a base ten numeral, such as 3284, written as $3(10^3) + 2(10^2) + 8(10^1) + 4$ was said to be in expanded form. Written as 3284 the numeral is said to be in positional notation. In base ten this form is also called decimal notation. Each digit represents a certain value according to its place in the numeral. In the above example, the 3 is in thousands place, the 2 is in hundreds place and so on. The form for place value in base ten shows that the value of each place immediately to the left of a given place is ten times the value of the given place. Each place value immediately to the right of a given place is one tenth of the place value of the given place. Looking back to the number 3284 written in expanded form you will notice that reading from left to right the exponents of ten decrease. Suppose we want to write 5634.728 in expanded form. We could write

$5(10^3) + 6(10^2) + 3(10^1) + 4 + 7(\ ?\ ) + 2(\ ?\ ) + 8(\ ?\ )$. The 4 is in unit's (or one's) place. The value of this place is $\frac{1}{10}$ of the value of the place before it. If we extend our numeral to the right of one's place and still keep this pattern, what will be the value of the next place? Of the next place after that one? To write this in expanded form to the right of unit's place as well as to the left of unit's place we have:

$5(10^3) + 6(10^2) + 3(10^1) + 4 + 7(\frac{1}{10}) + 2(\frac{1}{10^2}) + 8(\frac{1}{10^3})$ where the first place to the right of one's place is the one-tenth's place or simply tenth's place.

The following chart shows the place values both to the left and to the right of the unit's place.

Place Value Chart

Hundred thousand	Ten thousand	Thousand	Hundred	Ten	Unit or one	Tenth	Hundredth	Thousandth	Ten-thousandth	Hundred-thousandth
100,000	10,000	1,000	100	10	1	0.1	0.01	0.001	0.0001	0.00001
10^5	10^4	10^3	10^2	10^1	10^0	$\frac{1}{10}$	$\frac{1}{10^2}$	$\frac{1}{10^3}$	$\frac{1}{10^4}$	$\frac{1}{10^5}$

We usually refer to the places to the right of one's place as the decimal places.

When the number 3284.569 is written in decimal notation the decimal point locates the one's place. The number is read "Three thousand two hundred eighty-four and five hundred sixty-nine thousandths" or "Three two eight four -- point -- five six nine.

Example 1.

Write $5(10^2) + 7(10^1) + 3(\frac{1}{10})$ in decimal notation.

Notice that the one's place was not written. Could one's place be written $0(10^0)$? Your answer 570.3 will help you answer this question.

Exercises 9-3

1. Write each of the following in decimal notation:

   (a) $6(10) + 5(1) + 8(\frac{1}{10}) + 7(\frac{1}{10^2})$

   (b) $4(10^2) + 3(10) + 6(1) + 1(\frac{1}{10}) + 9(\frac{1}{10^2})$

   (c) $5(10) + 2(\frac{1}{10}) + 4(\frac{1}{10^2})$

   (d) $4(\frac{1}{10}) + 8(\frac{1}{10^2}) + 3(\frac{1}{10^3})$

   (e) $2(\frac{1}{10^3}) + 6(\frac{1}{10^4})$

   (f) $3(\frac{1}{10}) + 5(\frac{1}{10^5})$

   (g) $3(10^2) + 4(\frac{1}{10^2})$

2. Write each of the following in expanded form:

   (a) 52.55

   (b) 1.213

   (c) 0.4

   (d) 3.01

   (e) 0.0102

   (f) 0.10001

   (g) 30.03

3. Write each of the following in words: For example, 3.001 is written three and one thousandth.

(a)	7.236	(d)	1.0101
(b)	0.004	(e)	909.009
(c)	360.101	(f)	3.0044

4. Write in decimal numeral form:
   - (a) Three hundred and fifty-two hundredths
   - (b) Five hundred seven ten-thousandths
   - (c) Fourteen thousandths
   - (d) Sixty and 7 hundredths
   - (e) Thirty-two hundred-thousandths
   - (f) Eight and nineteen thousandths

*5. Write 0.1_{two} in base ten.

*6. Write $\frac{1}{2}$ in the duodecimal system.

*7. Write $\frac{5}{6}$ in the duodecimal system.

*8. Change 10.011_{two} to base ten.

9-4. Operations with Decimals

Suppose we wish to add two numbers in decimal form; for example, 0.73 + 0.84. We know how to add: 73 + 84 = 157. How do we handle the decimal places? We may proceed as follows:

$$0.73 = 73 \times \frac{1}{100} \text{ and } 0.84 = 84 \times \frac{1}{100}.$$

Therefore,

$$\begin{aligned} 0.73 + 0.84 &= (73 \times \frac{1}{100}) + (84 \times \frac{1}{100}) \\ &= (73 + 84) \times \frac{1}{100} \\ &= 157 \times \frac{1}{100} = 1.57. \end{aligned}$$

Notice that we have used the distributive property here.

Suppose now we wish to find the sum, 0.73 + 0.125. Just as before, we first rewrite our numbers as fractions:

$$0.73 = 73 \times \frac{1}{100} = 730 \times \frac{1}{1000} \text{ and } 0.125 = 125 \times \frac{1}{1000}.$$

We use the last form of 0.73, since then $\frac{1}{1000}$ appears in both products.

$$\begin{aligned} 0.73 + 0.125 &= \left(730 \times \frac{1}{1000}\right) + \left(125 \times \frac{1}{1000}\right) \\ &= (730 + 125) \times \frac{1}{1000} \\ &= 855 \times \frac{1}{1000} = 0.855. \end{aligned}$$

These examples can be handled more conveniently by writing one number below the other as follows:

$$\begin{array}{r} .73 \\ + \ .84 \\ \hline 1.57 \end{array} \qquad\qquad \begin{array}{r} 0.73 \\ + \ 0.125 \\ \hline 0.855 \end{array}$$

Notice that we write the decimal points directly under one another. This is because we want to add the number in the $\frac{1}{10}$ place in the first addend to the number in the $\frac{1}{10}$ place in the second addend, and the number in the $\frac{1}{100}$ place in the first addend to the number in the $\frac{1}{100}$ place in the second, etc. Thus,

$$0.73 = \frac{7}{10} + \frac{3}{100} \quad \text{and} \quad 0.84 = \frac{8}{10} + \frac{4}{100}$$

and therefore

$$\begin{aligned} 0.73 + 0.84 &= \frac{7 + 8}{10} + \frac{3 + 4}{100} \\ &= \frac{15}{10} + \frac{7}{100} \\ &= 1.57 \end{aligned}$$

Subtraction can be handled in the same way. For example,

$$\begin{array}{r} 0.84 \\ - \ 0.73 \\ \hline 0.11 \end{array} \qquad\qquad \begin{array}{r} 0.83 \\ - \ 0.74 \\ \hline 0.09 \end{array}$$

Exercises 9-4a

1. Add the following numbers.
   - (a) 0.76 + 0.84
   - (b) 0.719 + 0.382
   - (c) 1.002 + 0.0102
   - (d) 1.05 + 0.75 + 21.5

2. Subtract the following.
   (a) 0.84 - 0.76
   (b) 0.625 - 0.550
   (c) 0.500 - 0.125
   (d) 1.005 - 0.0005
3. Perform the operations indicated.
   (a) 1.051 - 0.702 + 0.066
   (b) 0.407 - 0.32 + 0.076
4. Four men enter a hardware store, and the first wants to buy 10.1 feet of copper wire, the second wants 15.1 feet, the third wants 8.6 feet, and the fourth wants 16.6 feet. The storekeeper has 50 feet of wire in his store. Can he give each man what he wants?
5. A storekeeper has 11.5 pounds of sugar. A woman buys 5.6 pounds. Another woman buys 4.8 pounds. Then a delivery truck brings 25 pounds to the store. Finally, mice eat 0.05 pounds. How much sugar is left?
6. There are 16 ounces in 1 pound. Which is heavier, 7 ounces or 0.45 lb.?
7. In most of Europe, distances are measured in kilometers (remember that a kilometer is about 5/8 of a mile). The distance from city A to Paris is 37.5 kilometers and the distance from city B to Paris is 113.2 kilometers. How far in kilometers is it from city A to city B by way of Paris?
8. Suppose the three cities, A, B and Paris in the previous problem are on the same road and city A is between Paris and city B. Then how far in kilometers is it between cities A and B along this road?

*9. Find the value of the sum, in the decimal system, of the numbers 10.01_{two} and 1.01_{two}. First change into the decimal system and then add.

Suppose we wish to multiply two numbers in decimal form. For example, 0.3 × 0.25. We know how to multiply these numbers: 3 × 25 = 75. Just as before, we write

$$\begin{aligned} 0.3 \times 0.25 &= 3 \times \frac{1}{10} \times 25 \times \frac{1}{100} \\ &= 3 \times 25 \times \frac{1}{10} \times \frac{1}{100} \\ &= 75 \times \frac{1}{1000} \\ &= 0.075 \end{aligned}$$

(1) How many digits are there to the right of the decimal point in 0.3?

(2) How many digits are there to the right of the decimal point in 0.25?

(3) What is the sum of the answers to (1) and (2)?

(4) How many digits are there to the right of the decimal point in 0.075?

(5) Compare the answers to (3) and (4).

Now multiply 0.4 × 0.25. What is your answer? Answer the five questions above, (1), (2), (3), (4), (5) for these numbers. Do the answers to (3) and (4) still agree?

Property 2: To find the number of decimal places in the product when two numbers are multiplied, add the number of decimal places in the two numerals.

For example, suppose we wish to multiply 0.732 by 0.25. The first numeral has three decimal places and the second has two, so there will be five decimal places in the answer. We find the product 732 × 25, and then mark off five decimal places in the answer, counting from right to left.

$$\begin{array}{r} .732 \\ \underline{.25} \\ 3660 \\ \underline{1464} \\ .18300 \end{array}$$

We consider now the problem of dividing one number in decimal form by another, for example, 0.125 ÷ 0.5.

The first step is usually to find a fraction whose denominator is a whole number and so that the new fraction is also a name for 0.125 ÷ 0.5. In this case, we start with $\frac{0.125}{0.5}$. Then, in order to replace the denominator by a whole number, we multiply numerator and denominator by 10 to get $\frac{1.25}{5}$. Using fractions we could work it out like this:

$$\frac{1.25}{5} = \frac{1}{5} \times 1.25 = \frac{1}{5} \times \left(\frac{1}{100} \times 125\right) = \frac{1}{100} \times \left(\frac{1}{5} \times 125\right)$$

$$= \frac{1}{100} \times 25 = 0.25$$

But a much shorter way is to use the usual form for division.

$$\begin{array}{r} 0.25 \\ 5\,\overline{)1.25} \\ \underline{1\;0} \\ 25 \\ \underline{25} \end{array}$$

Then:

(divisor)		(quotient)		(dividend)
5	×	0.25	=	1.25

and we see that in the equality the number of decimal places (two) in the product is the sum of the number of decimal places in the members of the product (0 + 2) just as we say in Property 2. Whenever the divisor is a whole number, the dividend and the quotient have the same number of decimal places. By placing the decimal point of the quotient directly above that of the dividend, we locate the decimal point of the quotient automatically in the correct place.

It is very easy to make a mistake in placing the decimal point of the answer and hence it is always a good plan to check by estimating the answer. For this example we see that

$$\frac{0.125}{0.5} \text{ is about } \frac{0.1}{0.5} = \frac{1}{5} = 0.2,$$

which is reasonably close to our product.

Try a more complicated fraction: $\frac{5.313}{2.53}$. If we multiplied the denominator by 10 we would have 25.3 which still is not a whole number. But if we multiply by 100, it becomes 253, which is a whole number. The given fraction then is equal to $\frac{531.3}{253}$ and we perform the division in the usual way:

$$
\begin{array}{rr}
 & 2.1 \\ \cline{2-2}
253 &)531.3 \\
 & 506 \\ \cline{2-2}
 & 25\;3 \\
 & 25\;3 \\ \cline{2-2}
\end{array}
$$

Then $253 \times 2.1 = 531.3$ and we see again that the number of decimal places in 253 (which is zero) plus the number of decimal places in 2.1 (which is 1) is equal to the number of decimal places in 531.3 (which is 1).

You should check the following:

$$\frac{0.75}{0.2} = \frac{7.5}{2} = \frac{7.50}{2} = 3.75\ .$$

Of course we cannot expect our divisions to "come out exactly" in all cases. Suppose we try to find the decimal expression for $\frac{2}{7}$. If we want the quotient to three decimal places, we will divide 2.000 by 7. If it were to be six places, we would divide 2.000000 by 7, and so forth. Suppose we find it to six places:

$$
\begin{array}{rl}
 & 0.285714\ldots \\ \cline{2-2}
7 &)2.000000 \\
 & 1\;4 \\ \cline{2-2}
 & 60 \\
 & 56 \\ \cline{2-2}
 & 40 \\
 & 35 \\ \cline{2-2}
 & 50 \\
 & 49 \\ \cline{2-2}
 & 10 \\
 & 7 \\ \cline{2-2}
 & 30 \\
 & 28 \\ \cline{2-2}
 & 2
\end{array}
$$

Exercises 9-4b

(Assume that the decimals are exact.)

1. Find the following products.
   - (a) 0.009×0.09
   - (b) 0.0025×2.5
   - (c) 1.2×120
   - (d) 0.135×0.202

2. Find the following quotients.
   (a) $0.009 \div 30$
   (b) $0.015 \div 0.05$
   (c) $0.575 \div 0.4$
   (d) $2.04 \div 0.008$

3. Express the following numbers as decimals.

   (a) $\frac{3000}{8}$ (b) $\frac{300}{8}$ (c) $\frac{30}{8}$ (d) $\frac{3}{8}$ (e) $\frac{3}{80}$ (f) $\frac{3}{800}$

4. $\frac{.015 \times .0025 \times 2.5}{.05 \times .03} = \ ?$

5. John's father has a garden 25.3 feet long and 15.7 feet wide. How many square feet are there in the garden?

6. A rectangle is 14.2 meters long and 5.7 meters wide. What is its area in square meters?

7. The dimensions of a box are: 17.3 meters by 8.3 meters by 2.5 meters. What is its volume in cubic meters?

8. About how many miles is a distance of 3.8 kilometers? (One kilometer is about $\frac{5}{8}$ of a mile.)

*9. Find the following product:

$$1.14_{\text{seven}} \times 2.4_{\text{seven}}$$

9-5. Decimal Expansion

Let us look more closely at rational numbers in decimal form. You recall that $\frac{1}{8}$ is the quotient of 1 divided by 8.

$$\begin{array}{r} 0.125 \\ 8\overline{)1.000} \end{array}$$

So $\frac{1}{8}$ and 0.125 are names for the same rational number.

Consider the rational number $\frac{1}{3}$. We all know the decimal numeral which names $\frac{1}{3}$. It is found by dividing 1 by 3.

$$\begin{array}{r} 0.333... \\ 3\overline{)1.000} \end{array}$$

Without performing the division do you know what digits will appear in each of the next 6 places? At some stage in the division do we get a remainder of zero?

The 3 dots indicate that the decimal numeral never ends.

Look again at the decimal numeral for $\frac{1}{8}$. Recall that it was obtained by dividing 1 by 8.

$$\begin{array}{r} 0.1250\ldots \\ 8\overline{)1.00000} \\ \underline{8} \\ 20 \\ \underline{16} \\ 40 \\ \underline{40} \\ 0 \\ \underline{0} \\ 0 \end{array}$$

After the first subtraction the remainder is 2.
The second remainder is 4.

The third remainder is 0.

The fourth remainder is 0. Can you predict the fifth and sixth remainders?

We usually stop our division process at the stage where we obtain a zero remainder. However we could just as well continue dividing, getting at each new stage a remainder of zero, and a quotient of zero.

It is clear that once we get a remainder of zero every remainder thereafter will be zero. We could write $\frac{1}{8} = 0.125000\ldots$ just as we wrote $\frac{1}{3} = 0.333\ldots$, but we seldom do. Therefore 0.125000... is a repeating decimal with 0 repeating over and over again. Likewise 0.333... is a repeating decimal with the digit 3 repeating.

By decimal expansion we mean that there is a digit for every decimal place. Note that the decimal expansion of $\frac{1}{8}$ and the decimal expansion of $\frac{1}{3}$ are both repeating.

Class Discussion Exercises

Let us look at the decimal which names the rational number $\frac{1}{7}$.

$$\begin{array}{r} 0.142857142857\ldots \\ 7\overline{)1.000000000000} \end{array}$$

1. Can you tell, without performing the division, the digits that should appear in each of the next six places?
2. Is there a block of digits which continues to repeat endlessly? Let us place a horizontal bar over the block of digits which repeats. Thus $0.142857\overline{142857}\ldots$uses the bar

to mean that the same digits repeat in the same order and the 3 dots to mean that the decimal never ends.

3. Name $\frac{1}{11}$ by a decimal numeral.
4. How soon can you recognize a pattern?
5. Will there be a zero remainder if you continue dividing?
6. Does this decimal repeat? How should you indicate this?
7. Observe that the decimal repeats as the remainder repeats. Look at the procedure by which we found a decimal numeral for $\frac{1}{7}$.

$$\begin{array}{l} \quad 0.142857142857\ldots \\ 7\overline{)1.000000000000} \\ \quad \underline{7} \\ \quad\ 30 \\ \quad\ \underline{28} \\ \quad\ \ 20 \\ \quad\ \ \underline{14} \\ \quad\ \ \ 60 \\ \quad\ \ \ \underline{56} \\ \quad\ \ \ \ 40 \\ \quad\ \ \ \ \underline{35} \\ \quad\ \ \ \ \ 50 \\ \quad\ \ \ \ \ \underline{49} \\ \quad\ \ \ \ \ \ 10 \\ \quad\ \ \ \ \ \ \underline{7} \\ \quad\ \ \ \ \ \ \ 30 \\ \quad\ \ \ \ \ \ \ \underline{28} \\ \quad\ \ \ \ \ \ \ \ 20 \\ \quad\ \ \ \ \ \ \ \ \underline{14} \\ \quad\ \ \ \ \ \ \ \ \ 60 \\ \quad\ \ \ \ \ \ \ \ \ \underline{56} \\ \quad\ \ \ \ \ \ \ \ \ \ 40 \\ \quad\ \ \ \ \ \ \ \ \ \ \underline{35} \\ \quad\ \ \ \ \ \ \ \ \ \ \ 50 \\ \quad\ \ \ \ \ \ \ \ \ \ \ \underline{49} \\ \quad\ \ \ \ \ \ \ \ \ \ \ \ 1 \end{array}$$

After the first subtraction the remainder is 3.

The second remainder is 2.

The third remainder is 6.

The fourth remainder is 4.

The fifth remainder is 5.

The sixth remainder is 1.

The seventh remainder is 3. Is the seventh digit to the right of the decimal point the same as the first?

The eighth remainder is the same as second. Is the eighth digit to the right of the decimal point the same as the second?

Notice that the digits in this quotient begin to repeat whenever a number appears as the remainder for the second time.

8. Make similar observations when you divide 1 by 37 to find a decimal numeral naming $\frac{1}{37}$.

Hence from these many illustrations you may conclude that every rational number can be named by a decimal numeral which either repeats a single digit or a block of digits over and over again.

Exercises 9-5

1. Write a decimal numeral (or decimal) for $\frac{1}{13}$.
    (a) How soon can you recognize a pattern?
    (b) Does this decimal end?
    (c) How should you indicate that it does not end?
    (d) Is there a set of digits which repeats periodically?
    (e) How should you indicate this?

2. Write decimals for:
    (a) $\frac{1}{3}$
    (b) $\frac{1}{4}$
    (c) $\frac{1}{6}$
    (d) $\frac{7}{8}$
    (e) $\frac{1}{9}$

    See how soon you can recognize a pattern in each case. In performing the division watch the remainders. They may give you a clue about when to expect the decimal numeral to begin to repeat.

3. Write the decimals for:
    (a) $\frac{1}{11}$
    (b) $\frac{2}{11}$
    (c) $\frac{3}{11}$
    (d) $\frac{9}{11}$
    (e) $\frac{14}{11}$
    (f) $\frac{23}{11}$

4. Study these decimal numerals and see if you can find a relationship
    (a) between the decimal naming $\frac{1}{11}$ and the decimal naming $\frac{2}{11}$;
    (b) between the decimal naming $\frac{1}{11}$ and the decimal naming $\frac{3}{11}$, $\frac{9}{11}$, $\frac{14}{11}$, etc.

5. Can you find a decimal for $\frac{5}{11}$ without dividing?

6. Is it true that the number $0.63\overline{63}\ldots$ is seven times the number $0.09\overline{09}\ldots$?

7. Find the decimal numeral for the first number in each group and calculate the others without dividing.

(a) $\frac{1}{5}, \frac{2}{5}, \frac{4}{5}$

(b) $\frac{1}{8}, \frac{3}{8}, \frac{7}{8}$

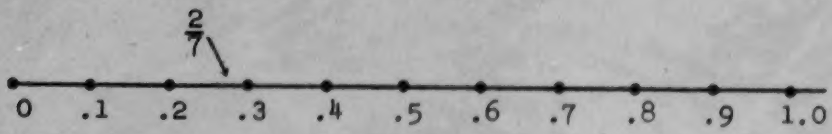
(c) $\frac{1}{20}, \frac{3}{20}, \frac{11}{20}$

(d) $\frac{1}{50}, \frac{42}{50}, \frac{47}{50}$

(e) $\frac{1}{1000}, \frac{111}{1000}, \frac{927}{1000}$

9-6. Rounding

Suppose we wish only an approximate decimal value for $\frac{2}{7}$. We might, for instance, want to represent this on the number line where each segment represents 0.1 as below:



Where would $\frac{2}{7}$ be on this line? We know its decimal form is 0.285714... . Perhaps you can see immediately from this that $\frac{3}{7}$ must lie between 0.20 and 0.30. If this is not clear, look at it a little more closely. First:

0.285714... = 0.2 + 0.08 + 0.005 + (other decimals), and hence it is certainly greater than 0.2, the first number in the sum. Second, 0.2 is less than 0.3; 0.28 is less than 0.30; 0.285 is less than 0.300 and so forth. No matter how far the decimal is computed, what we get is always less than 0.3000000... Also you can see that $\frac{2}{7}$ is closer to 0.30 than to 0.20 since 0.25 is halfway between 0.20 and 0.30 and the number 0.285714... is between 0.25 and 0.30 . So $\frac{2}{7}$ goes on the number line in about the place indicated by the arrow.

We could represent $\frac{2}{7}$ more closely on the number line if we divided each of the segments into ten parts. Then we would see that $\frac{2}{7}$ is between 0.28 and 0.29 but a little closer to 0.29. (You may want to write the reasons out in more detail than we did above, in order to make it clearer.) We would say that $\frac{2}{7}$ is 0.29 to the nearest hundredth. We call this "rounding $\frac{2}{7}$ to two places." What would $\frac{2}{7}$ be, rounded to three places?

The answer is: 0.286.

This rounding is useful in estimating results. For instance, suppose we have to find the product: 1.34×3.56. This would be approximately $1 \times 4 = 4$ or, if we wanted a little closer estimate, we could compute: $1.3 \times 3.6 = 4.7$ approximately.

Rounding is also useful when we are considering approximations in percents. For instance, if it turned out to be true that about 2 out of 7 families have dogs, it would be foolish to carry this out to many decimal places in order to get an answer in percent. We would usually just use two places and say that about 29% of all families have dogs, or we could round this still further and refer to 30%.

A special problem arises when the number to be rounded occurs exactly half-way between the two approximating numbers. For instance, how does one round 3.1215 to three decimal places? The two approximating numbers are 3.121 and 3.122 and one is just as accurate an approximation as the other. Often it makes little difference which decimal is used. However, if, for several numbers of a sum, one always rounded to the lower figure, the answer would probably be too small. For this reason, one often makes the agreement that he will choose the decimal whose last digit is even. Thus in the above case, we could choose 3.122 since its last digit is even; this number is larger than the given number. But if the given number were 3.1425, we would choose 3.142 as the approximating number since the given number lies between 3.142 and 3.143 and it is the decimal 3.142 whose last digit is even; here we have chosen the smaller of the two approximating numbers. This is not especially important for this class but you may find in your science class that this is what is done.

Exercises 9-6

1. Round the following numbers to two places.
    (a) 0.0351
    (b) 0.0449
    (c) 0.0051
    (d) 0.0193

2. Round the following numbers to three places.
   (a) 0.1599
   (b) 0.0009
   (c) 0.00009
   (d) 0.3249
3. Express the following numbers as decimals correct to three places.
   (a) $\frac{3}{8}$
   (b) $\frac{1}{4}$
   (c) $\frac{2}{3}$
4. Express the following numbers as decimals correct to one place.
   (a) $\frac{7}{23}$
   (b) $\frac{6}{23}$
   (c) $\frac{2}{23}$
   (d) $\frac{1}{23}$
5. (a) A piece of land is measured and the measurements are rounded to the nearest tenth of a rod (in other words, the measures are rounded to one decimal place). The length, after rounding is 11.1 rods and width is 3.9 rods. Find the area rounded to the nearest tenth of a square rod.
   (b) Suppose that the length is 11.14 rods and the width is 3.94 rods rounded to the nearest hundredth of a rod. Find the area rounded to the nearest hundredth of a square rod. What is the difference between this answer and the previous one?

9-7. Percents and Decimals

You have learned that the number $\frac{51}{100}$ can be written as 51% and also as 0.51. Actually we read both 0.51 and $\frac{51}{100}$ as "fifty-one hundredths". We have three different expressions for the same number:

$$\frac{51}{100} = 51\% = 0.51 ,$$

which we would read: "fifty-one hundredths is equal to fifty-one percent is equal to fifty-one hundredths." Similarly we have

$$\frac{1}{2} = \frac{50}{100} = 0.50 = 50\% = 0.5 .$$

Also $65\% = 0.65 = \frac{65}{100} = \frac{13}{20}$,

and $\frac{3}{5} = \frac{60}{100} = 0.60 = 60\% = 0.6$.

Class Exercises 9-7a

1. Express as percents:

   (a) $\frac{3}{25}$ (c) $\frac{73}{5}$

   (b) 0.4 (d) 1.2

2. Express as decimals:

   (a) 53% (c) $\frac{75}{4}$

   (b) 125% (d) 3%

It is a little more difficult to express $\frac{1}{8}$ as a percent since 8 is not a factor of 100. We know that its decimal form is 0.125. Other ways of writing this number are: $\frac{12.5}{100}$ or 12.5%; also $\frac{12\frac{1}{2}}{100}$ or $12\frac{1}{2}\%$. Similarly, 0.375 is the same as 37.5% or $37\frac{1}{2}\%$ and may also be written as:

$$\frac{375}{1000} = \frac{25 \cdot 15}{25 \cdot 40} = \frac{15}{40} = \frac{3 \cdot 5}{8 \cdot 5} = \frac{3}{8} .$$

Thus 0.375 is equal to both $37\frac{1}{2}\%$ and $\frac{3}{8}$.

Class Exercises 9-7b

1. Express as percents:

   (a) $\frac{5}{8}$ (c) $\frac{1}{125}$

   (b) $\frac{3}{16}$ (d) 0.475.

2. Express as decimals:

   (a) $62\frac{1}{2}\%$ (c) $16\frac{1}{4}\%$

   (b) $\frac{1}{125}$ (d) $\frac{3}{16}$.

How do we find the percent equivalent of $\frac{1}{3}$ whose decimal, 0.333... repeats endlessly? If we wish an approximate value we can round the decimal to two places and have:

$\frac{1}{3}$ is approximately equal to 0.33 or 33%.

An accurate name in percent for one-third can be found as follows:

$$\frac{1}{3} = \frac{\frac{1}{3}}{1} = \frac{\frac{1}{3} \cdot 100}{100} = \frac{33\frac{1}{3}}{100} = 33\frac{1}{3}\,\%$$

so that an accurate expression for $\frac{1}{3}$ is $33\frac{1}{3}\%$.

Class Exercises 9-7c

1. Express approximately as percents:

   (a) $\frac{2}{3}$ (c) $\frac{10}{9}$

   (b) $\frac{1}{9}$ (d) $\frac{2}{7}$

2. Express the above accurately as percents.

How can $28\frac{4}{7}\%$ be expressed as a fraction?

$$28\frac{4}{7}\% = \frac{28\frac{4}{7}}{100} = \frac{\frac{200}{7}}{100} = \left(\frac{200}{7} \times \frac{1}{100}\right) = \frac{2}{7} \quad .$$

Class Exercises 9-7d

Express the following as fractions:

(a) $66\frac{2}{3}\%$ (c) $25\frac{1}{4}\%$

(b) $11\frac{1}{9}\%$ (d) $125\frac{1}{2}\%$.

Exercises 9-7

1. Copy the following chart and fill in the missing names of numbers. The completed chart will be helpful to you in future lessons.

Fraction Simplest form	Hundred as denominator	Decimal	Percent
(a) $\frac{1}{2}$	$\frac{50}{100}$	0.50	50%
(b) $\frac{1}{4}$			
(c)	$\frac{75}{100}$		
(d)		0.20	
(e)			40%
(f)	$\frac{60}{100}$		
(g) $\frac{4}{5}$			
(h)		$0.3\overline{3}\ldots$	
(i)	$\frac{70}{100}$		
(j)		$0.6\overline{6}\ldots$	

Fraction Simplest form	Hundred as denominator	Decimal	Percent
(k) $\frac{3}{10}$			
(l)		0.10	
(m)			90%
(n) $\frac{1}{8}$			
(o)	$\frac{300}{100}$		
(p)		0.375	
(q)			150%
(r)	$\frac{62.5}{100}$		
(s)		0.01	
(t) $\frac{7}{8}$			
(u)			100%
(v)	$\frac{16\frac{2}{3}}{100}$		
(w) $\frac{5}{6}$			
(x) $\frac{1}{9}$			
(y)	$\frac{60\frac{1}{2}}{100}$		
(z)		0.005	

2. Draw a number line and name points on it with the percents in problem 1.
3. Using squared paper, draw a large square containing 100 small squares. By proper shading indicate the percents in parts (b), (d), (ℓ), (p), (s).

4. What fraction in simplest form is another name for
   (a) 32% (b) 90% (c) 120% ?

5. Give the percent names for the following numbers:

   (a) $\frac{13}{25}$ (b) $\frac{7}{20}$ (c) $\frac{19}{20}$ (d) $\frac{3}{10}$.

9-8. Applications of Percent

Percent is used to express ratios of numerical quantities in everyday experience. It is important for you to understand the notation of percent, and also, to be accurate in computing with numbers written as percents. Let us look at some examples in the use of percent.

Example 1. Suppose that a family has an annual income of $4500 (after withholding tax). The family budget includes an item for food of $33\frac{1}{3}\%$ of the budget. How much money is allowed for food for the year? Can you answer this question without using a pencil and paper? If you can, do so, and check what you find with the result below. We work this example as we do, to show a method which we shall be using in more difficult examples later. We know that $33\frac{1}{3}\%$ is equal to $\frac{1}{3}$ and hence if we let x stand for the number of dollars allowed for food we have

$$\frac{x}{4500} = \frac{1}{3} .$$

To find x we use Property 1 of Section 9-1 which tells us that

$$\text{if } \frac{a}{b} = \frac{c}{d} \text{ then } ad = bc.$$

For example, this means that

$$\text{if } \frac{6}{10} = \frac{9}{15} \text{ then } 6 \cdot 15 = 9 \cdot 10 .$$

Then in our example,

$$\text{if } \frac{x}{4500} = \frac{1}{3} \text{ then } 3x = 4500 .$$

Hence $x = \frac{4500}{3} = 1500$, and 1500 will be the number of dollars used for food.

Example 2. Suppose the annual income of the family in Example 1 is \$4560 and 32% of the budget is allowed for food for the year. Then if x stands for the number of dollars allowed for food, we have

$$\frac{x}{4560} = \frac{32}{100} .$$

Using Property 1, we have

$$100x = 32(4560) = 145,920 .$$

Hence

$$x = 1459.20$$

and the amount spent for food would be \$1459.20 which is close to the answer in Example 1.

Example 3. The number 900 is what percent of 1500? Can you find the answer to this example without use of pencil? This is a little harder than Example 1 but you may be able to solve it mentally. But the same method we used above, works here also. If $x\%$ stands for the percent which 900 is of 1500, we have

$$\frac{x}{100} = \frac{900}{1500} .$$

Again using Property 1, we have

$$1500x = 100(900) = 90000.$$

Hence

$$x = \frac{90,000}{1,500} = 60,$$

and 900 is 60% of 1500.

Example 4. Suppose a family with an annual income of \$4560 rents a house for \$77.00 per month. What percent of the family income will be spent for rent? First, the rent for the year will be twelve times the monthly rent, that is \$924.00. If $x\%$ is the percent that 924 is of 4560, we have, as before

$$\frac{x}{100} = \frac{924}{4560}$$

and, using Property 1 again, we have

$$4560x = 100(924)$$

$$4560x = 92,400$$

$$x = \frac{92,400}{4,560} = 20.3\ldots$$

Hence x% is about 20%; that is, about 20% of the family income is spent for rent.

Example 5. An advertisement said that a bicycle could be purchased "on time" by making a down payment of $14.70. The merchant stated further that this payment was 25% of the price. What was the price of the bicycle? If the price of the bicycle is x dollars, we wish to find x such that

$$\frac{14.70}{x} = \frac{25}{100} .$$

Before going on to finding x accurately, you should with your pencil and paper, estimate what x should be. Then, to find the accurate value of x, we use the same method as before to get:

$$100(14.70) = 25x$$

that is,

$$\frac{1470}{25} = x = 58.80.$$

So the price must have been $58.80.

Commissions and Discounts

People who work as salesmen often are paid a commission instead of a salary. A book salesman is paid a commission of 25% of the selling price of the set of books. If he sells a set of books for $60.00, his commission would be 25% of $60.00 or $15.00. Sometimes the percent of the selling price which gives the commission is called the rate of the commission.

Definition. Commission is the payment, often based on a percent of selling price, that is paid to a salesman for his services.

Merchants sometimes sell articles at a discount. During a sale, an advertisement stated "All coats will be sold at a discount of 30%." A coat marked $70.00 then sells at a discount of 30% of $70.00 or $21.00. The sale price (sometimes called the net price) is $70.00 - $21.00 or $49.00.

Definition. Discount is the amount subtracted from the marked price.

Definition. Sale price or net price is the marked price less the discount.

Exercises 9-8a

In the following problems it may be necessary to round some answers. Round the money answers to the nearest cent, and round percent answers to the nearest whole percent.

1. On an examination there was a total of 40 problems. The teacher considered all of the problems of equal value, and assigned grades by percent. How many correct answers are indicated by the following grades?
   (a) 100% (b) 80% (c) 50% (d) 65%
2. In grading the papers referred to in Problem 1 what percent would the teacher assign for the following papers?
   (a) All problems worked, but 10 answers are wrong.
   (b) 36 problems worked, all answers correct.
   (c) 20 problems worked, 2 wrong answers.
   (d) One problem not answered, and 1 wrong answer.
3. If the sales tax in a certain state is 4% of the purchase price, what tax would be collected on the following purchases?
   (a) A dress selling for $17.50
   (b) A bicycle selling for $49.50
4. A real estate agent receives a commission of 5% for any sale that he makes. What would be his commission on the sale of a house for $17,500?
5. The real estate agent referred to in Problem 4 wishes to earn an annual income from commissions of at least $9000. To earn this income, what would his yearly sales need to total?
6. A salesman who sells vacuum cleaners earns a commission of $25.50 on each one sold. If the selling price of the cleaner is $85.00 what is the rate of commission for the salesman?
7. Sometimes the rate of commission is very small. Salesmen for heavy machinery often receive a commission of 1%. If, in one year, such a salesman sells a machine to an industrial plant for $658,000 and another machine for $482,000, has he earned a good income for the year?
8. A sports store advertised a sale of football equipment. The discount was to be 27%.

(a) What would be the sale price of a football whose marked price is $5.98?

(b) What would be the sale price of a helmet whose marked price is $3.40?

9. In a junior high school there are 380 seventh-grade pupils, 385 eighth-grade pupils, and 352 ninth-grade pupils.
   (a) What is the total enrollment of the school?
   (b) What percent of the enrollment is in the seventh grade?
   (c) What percent of the enrollment is in the eighth grade?
   (d) What percent of the enrollment is in the ninth grade?
   (e) What is the sum of the numbers represented by the answers to (b), (c), and (d)?

10. Mr. Martin keeps a record of the amounts of money his family pays in sales tax. At the end of one year he found that the total was $96.00 for the year. If the sales tax rate is 4%, what was the total amount of taxable purchases made by the Martin family during the year?

11. Mr. Brown asked his bank for a loan of $1000 and promised to pay it back in a year. For this the bank charged him 6% of the $1000. This is called the "rate of interest." What was the amount he had to pay in interest? If he repaid the amount of the loan at the end of the year together with the interest, what did he pay the bank?

12. Mr. Jacobs arranged a loan from his bank of $2000. At the end of a year he paid to the bank $2140. What was the rate of interest? What percent is $2140 of $2000?

*13. A government bond which costs $18.75 is worth $25.00 ten years after it is purchased. By what percent did it increase in value over the ten years? What was the average percent of increase per year?

14. The real estate tax is $12 on a house valued at $1000, $24 on a house valued at $2000, $36 on a house valued at $3000. What percent of the value of the houses is the tax in each case? If the percent is the same, what would be the tax on a house valued at $25,000 ?

*15. A certain store gives a 10% discount for cash and a 5% discount for purchases made on Mondays. That is, if a customer purchased an article priced at $100 and paid cash it would cost him $100 - $10 = $90 . Then if the day of his purchase were Monday he would get a further 5% discount, which would make the net price $85.50 since 5% of 90 is 4.50 and

$$90 - 4.50 = 85.50 .$$

Suppose the 5% discount on $100 had been computed first and the 10% second, would the final net price be the same? Would these two ways of computing the final net price give the same result for an article priced at $200? Why?

*16. In a certain store each customer pays a sales tax of 2% and is given a 10% discount for cash. That is, if a customer purchased an article priced at $100 and paid cash it would cost him $90 plus the sales tax or $91.80, since 2% of $90 is $1.80 . Suppose the sales tax were computed on $100 and then the 10% discount allowed, would the resulting net cost be the same? Why or why not?

*17. A customer in the store of Problem 15 added the discounts and thought that since he was paying cash for an article on Monday, he should receive 15% discount. If this were the case he would have paid $85. for the article priced at $100, instead of the $85.50 . How should the shopkeeper have worded his notice of discounts to make it clear that he had in mind the calculation given in Problem *15?

Percents Used for Comparison

In Problem 1 of Exercises 9-7, some of the percents were not whole percents. The number $\frac{1}{8}$ written as a percent is $12\frac{1}{2}\%$. You have learned that $\frac{1}{2}\%$ can be read "$\frac{1}{2}$ of 1%." Decimals are often used in percents, for example, 0.7%.

$$0.7\% \,(0.7 \text{ of } 1\%) = 0.7 \times \frac{1}{100} = \frac{0.7}{100} = \frac{7}{1000} = 0.007$$

These are all names for the same number. If we wish to find 0.7%

of \$300, we wish to find x such that

$$\frac{x}{300} = \frac{0.7}{100} \qquad \text{or} \qquad \frac{x}{300} = 0.007$$

$$100x = 210 \qquad\qquad x = 300\ (0.007)$$

$$x = 2.10\ (\$2.10) \qquad\qquad x = 2.10\ (\$2.10)$$

0.7% is less than 1%. Since 1% of \$300 is \$3.00, the answer \$2.10 is reasonable.

Suppose that we wish to find 2.3% of \$500.

$2.3\% = \frac{2.3}{100} = \frac{23}{1000} = 0.023$. Find the number such that

$$\frac{x}{500} = \frac{2.3}{100} \qquad \text{or} \qquad \frac{x}{500} = 0.023$$

$$100x = 1150 \qquad\qquad x = 500\ (0.023)$$

$$x = 11.50\ (\$11.50) \qquad\qquad x = 11.50\ (\$11.50)$$

Baseball batting averages are found for each player by dividing the number of hits he has made by the number of times he has been at bat. The division is usually carried to the nearest thousandth. So the batting average can be considered as a percent expressed to the nearest tenth of a percent. If a player has 23 hits out of 71 times at bat, his batting average is $\frac{23}{71}$ or .324.

Sometimes answers are called for to the nearest tenth of a percent. A teacher issues 163 grades, 35 of them B. He might be asked what percent of his grades are B. He wishes to find x such that $\frac{35}{163} = \frac{x}{100}$.

$$\frac{35}{163} = \frac{x}{100} \qquad \text{or} \qquad \frac{35}{163} = \frac{x}{100}$$

$$x = \frac{35}{163} \cdot 100 \qquad\qquad 163x = 3500$$

$$x = 21.47\ldots \qquad\qquad x = 21.47\ldots$$

In section 9-6 you learned how to round decimals. If the percent is called for to the nearest tenth of a percent, 21.47... would round to 21.5%.

Percents of Increase and Decrease

Percent is used to indicate an increase or a decrease in some quantity. Suppose that Central City had a population of 32,000 (rounded to the nearest thousand) in 1950. If the population increased to 40,000 by 1960, what was the percent of increase?

$$\begin{array}{r} 40,000 \\ -\ \underline{32,000} \\ 8,000 \end{array} \text{ (actual increase)}$$

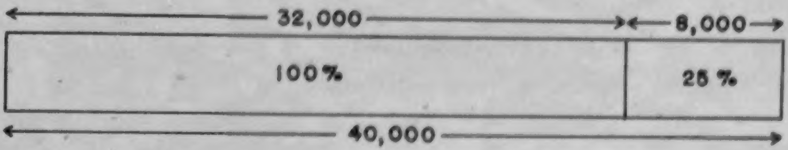
$$\frac{8,000}{32,000} = \frac{x}{100}$$

$$x = \frac{800,000}{32,000}$$

$$x = 25$$

There was an increase of 25%.

Notice that the percent of increase is found by comparing the actual increase with the earlier population figure.



The 40,000 was made up of the 32,000 (100%) plus the increase of 8,000 (25%). So the population of 40,000 in 1960 was 125% of the population of 32,000 in 1950.

Suppose that Hill City had a population of 15,000 in 1950. If the population in 1960 was 12,000, what was the percent of decrease? If x represents the percent of decrease, then

$$\begin{array}{r} 15,000 \\ -\ \underline{12,000} \\ 3,000 \end{array} \text{ actual decrease}$$

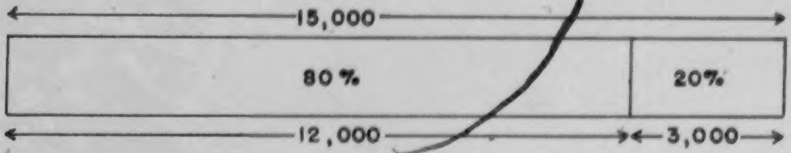
$$\frac{3,000}{15,000} = \frac{x}{100}$$

$$x = 100 \cdot \frac{3,000}{15,000}$$

$$x = 20$$

There was a decrease of 20%.

Notice that the population decrease also is found by comparing the actual decrease with the earlier population figure.



The 12,000 is the difference between 15,000 (100%) and the decrease of 3,000 (20%). So the population in 1960 of 12,000 was 80% of the population of 15,000 in 1950.

If the rents in an apartment house are increased 5%, each tenant can compute his new rent. Suppose that a tenant is paying $80 for rent, what will he pay in rent after the increase? If x represents the increase in number of dollars, then

$$\frac{x}{80} = \frac{5}{100}$$

$$x = 80 \cdot \frac{5}{100}$$

$$x = 4 \quad \text{(The increase was \$4.00.)}$$

The new rent will be $80 + $4 = $84.

Exercises 9-8b

1. A junior high school mathematics teacher had 176 pupils in his 5 classes. The semester grades of the pupils were A, 20; B, 37; C, 65; D, 40; E, 14.
   (a) What percent of the grades were A? (to nearest tenth percent)
   (b) What percent of the grades were B?
   (c) What percent of the grades were C?
   (d) What percent of the grades were D?
   (e) What percent of the grades were E?
   (f) What is the sum of the answers in parts (a), (b), (c), (d), (e)? Does this sum help check the answers?
2. Bob's weight increased during the school year from 65 pounds to 78 pounds. What was the percent of increase?
3. During the same year, Bob's mother reduced her weight from 160 pounds to 144 pounds. What was the percent of decrease?
4. The enrollment in a junior high school was 1240 in 1954. By 1960 the enrollment had increased 25%. What was the enrollment in 1960?
5. Jean earned $14.00 during August. In September she earned only $9.50. What was the percent of decrease in her earnings?

6. A salesman of heavy machinery earned a commission of $4850 on the sale of a machine for $970,000.
   (a) At what rate is his commission paid?
   (b) What will be his commission for the sale of another machine for $847,500?
7. James was 5 ft. tall in September. In June his height is 5 ft. 5 in. Both heights were measured to the nearest inch. What is the percent of increase in height?
8. Do you know what your height was at the beginning of the school year? Now? Do you know what your weight was at the beginning of the school year? Now?
   (a) What is the percent of increase in your height since last September?
   (b) What is the percent of increase in your weight since last September?
9. A baseball player named Jones made 25 hits out of 83 times at bat. Another player named Smith made 42 hits out of 143 times at bat.
   (a) What is the batting average of each player?
   (b) Which player has the better record?
10. An elementary school had an enrollment of 790 pupils in September, 1955. In September, 1959, the enrollment was 1012. What was the percent of increase in enrollment?
11. On the first of September Bob's mother weighed 130 pounds. During that month she decreased her weight by 15%. However, during the month of October her weight increased by 15%. What did she weigh on the first of November? Are you surprised at your answer? Why or why not?
12. Suppose in the previous problem the weight of Bob's mother increased by 15% during the month of September and decreased by 15% during the month of October. Would the result be the same as in the previous problem? Can you see why or why not? Then check your guess by working it out.

Two Methods for Solving Problems of Percent of Increase and Decrease

In the solution of problems involving percent of increase or percent of decrease, two approaches can be used.

If the cost of butter increases from 80¢ a pound to 92¢ a pound, what is the percent of increase? The method you have been using is

92¢ - 80¢ = 12¢ (increase). If 12 is x percent of 80, then

$$\frac{12}{80} = \frac{x}{100}$$

$$x = 100 \cdot \frac{12}{80}$$

$$x = 15$$

There was a 15% increase.

A second method finds what percent 92¢ is of 80¢. Since 92 is larger than 80, then 92 is more than 100% of 80. If 92 is x percent of 80, then

$$\frac{92}{80} = \frac{x}{100}$$

$$x = 100 \cdot \frac{92}{80}$$

$$x = 115$$

This means that 92¢ is 115% of 80¢. If 100% is subtracted from 115%, the percent of increase (15%) results.

In a certain city the fire department extinguished 160 fires during 1958. During 1959, the number of fires extinguished dropped to 120. What was the percent of decrease? We will show two ways to solve this problem. In one method we find what percent the difference (160 - 120) is of 160. In the other method we find what percent the number of fires in the later year is of the number of fires in the earlier year. This percent is then compared with 100%. If 120 is y percent of 160, then

$$\begin{array}{r} 160 \\ -\underline{120} \\ 40 \end{array}$$ (fewer fires)

if 40 is x percent of 160, then

$$\frac{40}{160} = \frac{x}{100}$$

$$x = 100 \cdot \frac{40}{160}$$

$$x = 25$$

$$\frac{120}{160} = \frac{y}{100}$$

$$y = 100 \cdot \frac{120}{160}$$

$$y = 75$$

100% - 75% = 25%

The later number of fires was 75% of the earlier number. There was a decrease of 25% for the number of fires in 1959 when compared with the number in 1958.

There was a decrease of 25% in the number of fires in 1959 when compared with the number in 1958.

Of course, the answers should be the same for the two methods of solution.

Exercises 9-8c

In each problem, 1 through 5, compute the percent of increase or decrease by both methods. If necessary, round percents to the nearest tenth of a percent.

1. In a junior high school the lists of seventh-grade absentees for a week numbered 29, 31, 32, 28, 30. The next week the five lists numbered 22, 26, 24, 25, 23.
   (a) What was the total number of pupil-days of absence for the first week?
   (b) What was the total for the second week?
   (c) Compute the percent of increase or decrease in the number of pupil-days of absence.
2. On the first day of school a junior high school had an enrollment of 1050 pupils. One month later the enrollment was 1200. What was the percent of increase?
3. One week the school lunchroom took in $450. The following week the amount was $425. What was the percent of decrease?

4. From the weight at birth, a baby's weight usually increases 100% in six months.
   (a) What should a baby weigh at six months, if its weight at birth is 7 lb. 9 oz.?
   (b) Suppose that the baby in (a) weighs 17 lb. at the age of six months. What is the percent of increase?

5. During 1958 a family spent $1490 on food. In 1959 the same family spent $1950 on food. What was the percent of increase in the money spent for food?

*6. During 1958 the owner of a business found that sales were below normal. The owner announced to his employees that all wages for 1959 would be cut 20%. By the end of 1959 the owner noted that sales had returned to the 1957 levels. The owner then announced to the employees that the 1960 wages would be increased 20% over those of 1959.
   (a) Which of the following statements is true?
      (1) The 1960 wages are the same as the 1958 wages.
      (2) The 1960 wages are less than the 1958 wages.
      (3) The 1960 wages are more than the 1958 wages.
   (b) If your answer to part (a) is (1), justify your answer. If your answer to part (a) is (2) or (3) express the 1960 wages as a percent of 1958 wages.

*7. In an automobile factory the number of cars coming off the assembly line in one day is supposed to be 500. One week the plant operated normally on Monday. On Tuesday there was a breakdown which decreased the number of completed cars to 425 for the day. On Wednesday operations were back to normal.
   (a) What was the percent of decrease in production on Tuesday compared with Monday?
   (b) What was the percent of increase in production on Wednesday compared with Tuesday?

8. (a) A salesman gets a 6% commission on an article which he sells for $1000. How much commission does he get?
   (b) A bank gets 6% interest per year on a loan of $1000. How much interest does the bank get?

(c) The tax on some jewelry is 6% . How much tax would one have to pay on a pearl necklace worth $1000.?

(d) Can you find any relationship among (a), (b), and (c) ?

9. (a) An article is sold at a 5% discount. If the stated price is $510, what did it sell for?

(b) A small town had a population of 510 people on January first, 1958. The population decreased 5% during the following twelve months. What was its population on January first, 1959?

(c) On a loan of $510, instead of charging interest, a bank loaned it on a discount of 5%; that is, gave the customer $510 minus 5% of $510 at the beginning of the year with the understanding that the amount of $510 would be paid back at the end of the year. How much did the customer receive at the beginning of the year?

(d) Can you find any relationship among (a), (b), and (c) ?

10. What was the interest rate in problem 9 (c) above?

11. (a) Mr. Brown paid $210 in gasoline taxes during a year. If the tax on gasoline is 31% , how much did he spend on gasoline?

(b) Mr. Smith made a down payment of $210 on a washing machine. If this was 31% of the total cost, what did the washing machine cost?

(c) In a certain town 31% of the population was children. If there were 210 children, what was the population of the town?

(d) On an article priced at $210, a merchant made a profit of 31% . What was the amount of his profit in dollars?

(e) Can you find any relationship among (a), (b), (c), and (d)?

*12. The income tax collector looked at the income tax return of Mr. Brown mentioned in Problem 11 (a). He inquired to find that Mr. Brown drove a Volkswagen which would go about 30 miles on a gallon of gasoline. He also found that Mr. Brown could walk to work.

(a) If gasoline cost $.30 a gallon (including tax), how many gallons did Mr. Brown buy? (Use information from Problem 11 (a).)

(b) How far could he drive with this amount of gasoline?

(c) What would be the average per day?

(d) Why did the tax collector question Mr. Brown's return?

Chapter 10

PARALLELS, PARALLELOGRAMS, TRIANGLES, AND RIGHT PRISMS

10-1. Two Lines in a Plane

As early as 4000 B.C. geometry had an influence on the way in which man lived. Clay tablets made in Babylon some 6000 years ago show that Babylonians knew how to find the area of a rectangular field. They used the same relation you learned in Chapter 8.

The great pyramid at Giza in Egypt was built about 2900 B.C. The constructions of this and other pyramids show that the Egyptians too must have known much about geometry. About 1850 B.C., Egyptian manuscripts were being written which show that the Egyptians were developing geometric rules. One of these was the general rule for finding the area of a triangle, which will be discussed later in this chapter.

When a mathematician begins an investigation, he usually starts with a very simple case. After he feels that he understands this case, he may then proceed to more complicated situations. In order to get a feeling for space relationships, let us begin by studying more about figures formed by two lines.

Figure 10-1-a shows that the intersection of lines ℓ_1 and ℓ_2 is point A. Rays on ℓ_2 are $\overrightarrow{AB}$ and $\overrightarrow{AC}$. (Recall that $\overrightarrow{AB}$ means the ray with A as endpoint and containing point B). Rays on ℓ_1 are $\overrightarrow{AE}$ and $\overrightarrow{AD}$.

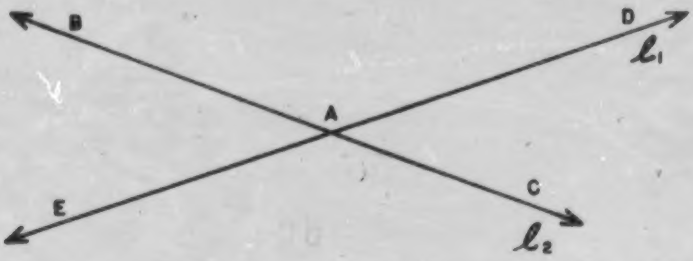


Figure 10-1-a

Observe that there are four angles formed by the rays in Figure 10-1-a: angle BAE, angle BAD, angle CAD, and angle CAE. We shall speak of these angles as angles formed by ℓ_1 and ℓ_2. In this chapter, then, we shall speak of two intersecting lines as determining four angles.

Look at angles CAD and DAB. These two angles have a common ray, AD, and a common vertex, point A. Any two angles which have a common ray, a common vertex, and whose interiors have no point in common are called adjacent angles. (Adjacent means "neighboring".) Thus, angles CAD and DAB are adjacent angles. Are there any more pairs of adjacent angles in the figure? Yes: DAB and BAE, BAE and EAC, EAC and CAD, are all pairs of adjacent angles.

Are angles BAD and EAC adjacent angles? No, they are not! But they are both formed by rays on the two lines ℓ_1 and ℓ_2. In this chapter, when we speak of a line we will always mean a straight line, extending without end in both directions. When two lines intersect, the two pairs of non-adjacent angles formed by these lines are given a special name, vertical angles. Note that here "vertical" is not associated with "horizontal." Thus, angles BAE and CAD are a pair of vertical angles.

In Figure 10-1-a, consider the line $\overleftrightarrow{BC}$ containing the rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$. Recall that in Chapter 7 you learned that if we place a protractor with vertex at A so that $\overrightarrow{AC}$ is the zero ray, then $\overrightarrow{AB}$ corresponds to the number 180 on a protractor.

A line divides a plane into two half-planes. In Figure 10-1-b, one half-plane determined by ℓ_1 is shown by the shaded area above ℓ_1. The other half-plane determined by ℓ_1 is shown as the non-shaded area below ℓ_1.

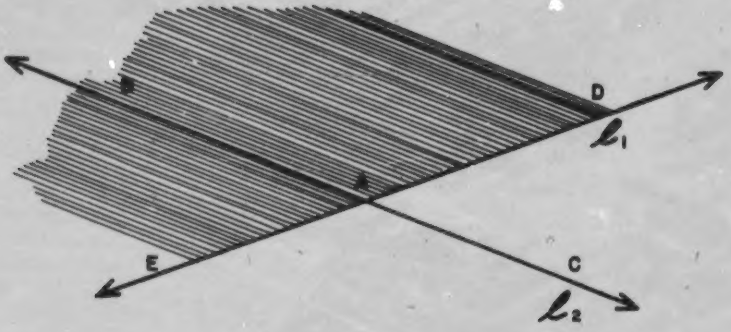


Figure 10-1-b

In Figure 10-1-c, the two half-planes determined by ℓ_2 are shown by the shaded area below ℓ_2 and the non-shaded area above ℓ_2.

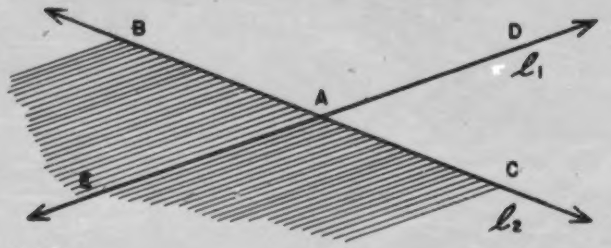


Figure 10-1-c

Thus, two intersecting lines separate a plane into four regions. Each of these four regions is the interior of an angle. The interior of each angle is the intersection of two half-planes. The intersection of the shaded half-plane in Figure 10-1-b and the shaded half-plane in Figure 10-1-c is shown in Figure 10-1-d. The intersection of these two half-planes is the interior of angle BAE. The interior of angle CAD is the intersection of the two remaining unshaded half-planes. These two angles, angle BAE and angle CAD, are vertical angles. Can you see the two half-planes which include the interior of angle CAD? Can you see the two half-planes which include angle BAD?

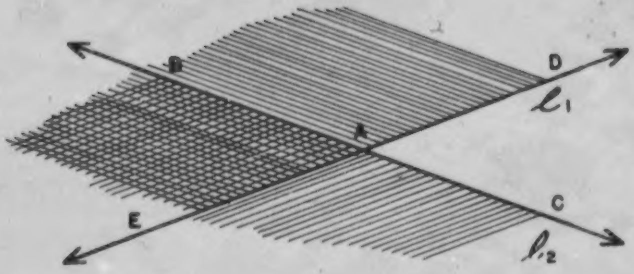


Figure 10-1-d

If the sum of the measures in degrees of two angles is 180, the angles are called supplementary angles. In Figure 10-1-d, the shaded angles BAE and BAD are supplementary. Are any other pairs of angles in this figure supplementary? Are ∠ CAD and ∠ BAD supplementary angles? What about ∠ CAE and ∠ CAD? In Figure 10-1-a the supplementary angles are also adjacent angles. Angles BAE and BAD are adjacent angles, as are angles CAD and BAD. In Figure 10-1-e, if the measure of angle M is 40 and the measure of angle N is 140, then angles M and N are supplementary angles.

Figure 10-1-e

You learned in Chapter 7 how to measure angles. We shall frequently talk about the "measure of an angle" in this chapter. Therefore, it is convenient to have a symbol for this statement. To indicate the number of units in an angle, we will use the symbol "m" followed by the name of the angle enclosed in parentheses. For example, $m(\angle ABC)$ means the number of units in angle ABC.

As you learned, any angle can be used as a unit of measure, but in this chapter we will use the degree as the standard unit. Thus when we write $m(\angle ABC) = 40$, we will understand angle ABC is a 40 degree (40°) angle. Note that since $m(\angle ABC)$ is a number, we write only $m(\angle ABC) = 40$, not "$m(\angle ABC) = 40^\circ$."

Exercises 10-1

Use Figure 10-1-f in answering 1 through 3.

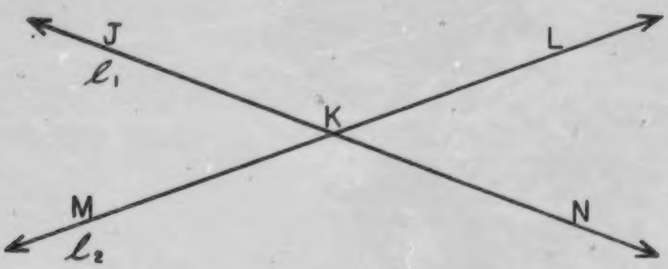


Figure 10-1-f

1. (a) Name the angles adjacent to $\angle$ JKM.
   (b) Name the angles adjacent to $\angle$ LKN.
   (c) Name the angles adjacent to $\angle$ JKL.
2. (a) Name the angle which with $\angle$ JKM completes a pair of vertical angles.
   (b) Name another pair of vertical angles in the figure.
   (c) When two lines intersect in a point, how many pairs of vertical angles are formed?

3. (a) Use a protractor to find the measures of the vertical angles, $\angle$ JKM and $\angle$ LKN.

   (b) What are the measures of angles NKM and JKL?

   (c) What appears to be true concerning the measures of a pair of vertical angles?

   (d) Draw sets of two intersecting lines as in Figure 10-1-f. Vary the size of the angles between the lines. With a protractor find the measures of each pair of vertical angles. Do they appear equal?

4. Study your answers to Problem 3. Now state your results in the form of a general property by copying and completing the following sentence:

   Property 1: When two lines intersect, the two angles in each pair of ___?___ angles which are formed have ___?___ measure, or are congruent.

*5. You have found by experiment a certain relation to be true in a number of cases. Let us see why this relation must be true in all cases.

   (a) In Figure 10-1-f what is the sum of the measures of $\angle$ JKL and $\angle$ JKM?

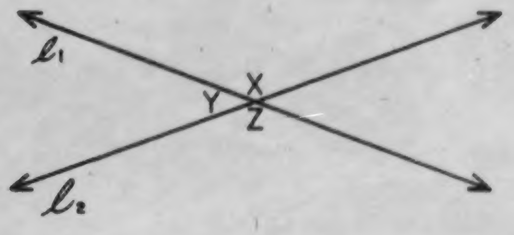
   (b) What is the sum of the measures of $\angle$ NKM and $\angle$ JKM?

   (c) If the measure of $\angle$ JKM is 60, then what is the measure of $\angle$ JKL? Of $\angle$ NKM?

   (d) If the measure of angle JKM is 70, then what is the measure of $\angle$ JKL? Of $\angle$ NKM?

   (e) What can you say about the measures of angles JKL and NKM when the measure of angle JKM changes? Explain why angles JKL and NKM in Figure 10-1-f must have the same measure.

*6. The following figure is similar to Figure 10-1-f. Let x, y, and z represent the angles LKJ, JKM, and NKM respectively. The angles are indicated in this way in the figure.



Copy and complete the following statements:

(a) m(∠ x) + m(∠ y) = __?__.

(b) m(∠ z) + m(∠ y) = __?__.

(c) If m(∠ y) is known, how can you find m(∠ x)? How can you find m(∠ z)?

(d) Write your answer for part (c) in the form of a number sentence as is done in parts (a) and (b) by copying and completing the following:

m(∠ x) = __?__ - __?__
m(∠ z) = __?__ - __?__

(e) Write a number sentence to show the relation between m(∠ x) and m(∠ z).

*7. Imagine two lines in space.

(a) Is there any possible relation between two lines in space which would not occur between two lines in a plane?

(b) Find an illustration in your classroom to explain your answer.

10-2. Three Lines in a Plane

The Greeks were the first to study geometry as a field of knowledge. Thales (640 B.C. - 546 B.C.) studied in Egypt and introduced the study of geometry into Greece. The discovery of the property dealing with pairs of vertical angles which we studied in the previous section is credited to Thales. As you study geometry you will learn about properties that he and other Greek mathematicians discovered.

In the previous section we primarily studied figures formed by two lines in a plane. In this section we will study figures formed by three lines in a plane.

Draw a figure similar to 10-1-a. Can you draw another line, ℓ_3, through point A? Can lines ℓ_1, ℓ_2, and ℓ_3 have a common point of intersection? Your drawing should look something like this:

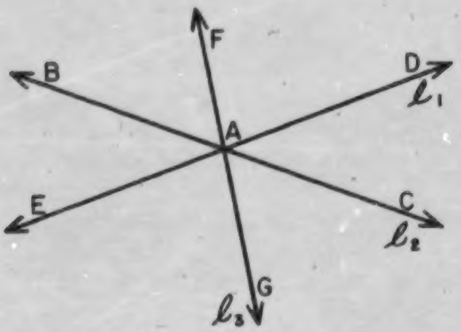


Figure 10-2-a

Will three lines drawn on a plane always have a common point of intersection? Look at Figure 10-2-b where line t intersects lines ℓ_1 and ℓ_2. In the language of sets we would say $\ell_1 \cap t$ is not the empty set, (and $\ell_2 \cap t$ is not the empty set). A line which intersects two or more lines in distinct points is called a transversal of those lines. Since line t intersects ℓ_1 and ℓ_2, t is a transversal of lines ℓ_1 and ℓ_2.

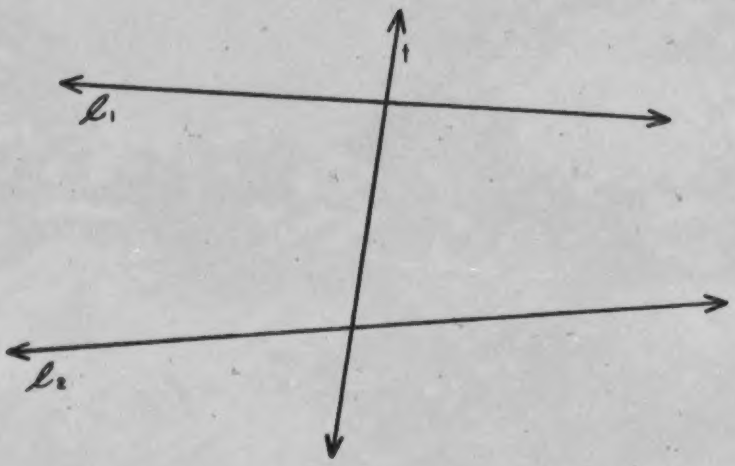
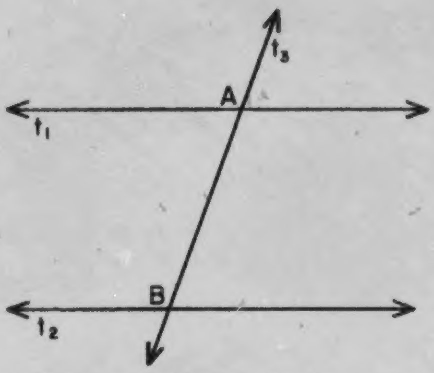


Figure 10-2-b

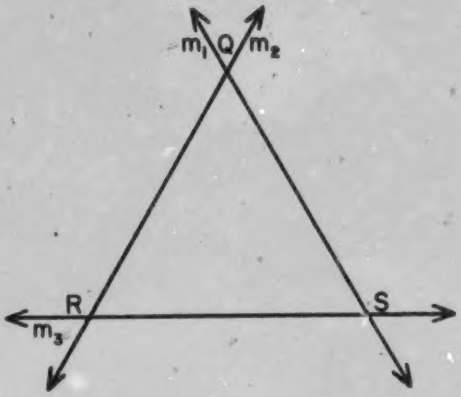
Exercises 10-2



1. Draw a figure similar to the one at the right. Lines t_1 and t_2 do not intersect. ($t_1 \cap t_2$ is the empty set). Line t_3 intersects t_1. Call the point of intersection A. Line t_3 intersects t_2. Call the point of intersection B.
   - (a) How many pairs of vertical angles are there in the figure you have drawn?
   - (b) How many pairs of adjacent angles are there in the figure you have drawn?
   - (c) Is line t_3 a transversal of lines t_1 and t_2?
   - (d) Is t_1 a transversal of t_2 and t_3? Explain.

2. Draw two lines, m_1 and m_2, which intersect in point Q. Draw a third line, m_3, which intersects m_1 and m_2, but not in point Q. Call the intersection of m_1 and m_3 point S, and call the intersection of m_2 and m_3 point R.

   (a) What is the name of the set of points made up of segments $\overline{SQ}$, $\overline{QR}$, and $\overline{RS}$ in the figure you have drawn? (See Chapter 4 if you don't know.)



   (b) How many pairs of vertical angles are there in the figure you have drawn?

   (c) How many pairs of adjacent angles are there in the figure you have drawn?

3. Use Figure 10-2-c in answering the following questions:

   (a) Name the transversal shown in the drawing and tell what lines it intersects.

   (b) In this drawing, how many angles are formed by the three lines?

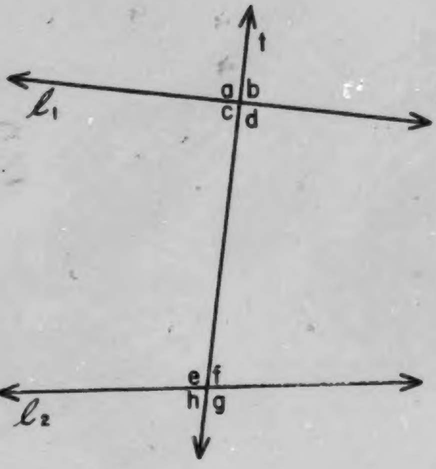


Figure 10-2-c

(c) How many pairs of vertical angles are there in this drawing?

(d) What do you know about the measures of each of a pair of vertical angles?

(e) Does $m(\angle h) = m(\angle f)$? How can you tell?

(f) Is $\angle c$ congruent to $\angle d$? How can you tell?

4. (a) How many pairs of adjacent angles are there in Figure 10-2-c?

(b) What do you know about the sum of the measures of any pair of adjacent angles in Figure 10-2-c?

5. (a) Are angles c and d in Figure 10-2-c supplementary?

(b) Are angles a and d in Figure 10-2-c supplementary?

(c) Are the angles in each pair of adjacent angles in Figure 10-2-c supplementary?

(d) If the measure of $\angle h$ is 80, what is the measure of $\angle g$? Of $\angle e$? Of $\angle f$?

(e) If the measure of $\angle h$ is 90, are angles h and f supplementary?

(f) If $m(\angle h) = m(\angle g)$, are angles e and g supplementary?

6. Look at the two angles, angle b and angle f, in Figures 10-2-d at the right. From the vertex of angle f there is a ray which extends upward on line t. This ray contains a ray of angle b. Also, the interiors of angle b and angle f are on the same side of the transversal t. Angles placed in this way are called corresponding angles.

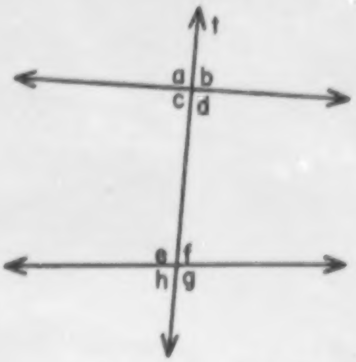


Figure 10-2-d

(a) Name another pair of corresponding angles on the same side of the transversal as angles b and f.

(b) Are angles a and e corresponding angles? (Note that the ray on line t which forms angle a is only a part of the ray on line t which forms angle e; however, neither of the rays forming angle e are parts of the rays forming angle a.)

(c) Are $\angle c$ and $\angle g$ corresponding angles?

(d) How many pairs of corresponding angles are in Figure 10-2-c?

(e) If the measure of $\angle b$ is 80, can you tell what the measure of $\angle f$ is?

(f) If the measures of $\angle a$ and $\angle e$ are 90, what can you say about the measures of all the angles in Figure 10-2-c?

(g) If the measures of $\angle a$ and $\angle e$ are 90, are angles h and b supplementary angles?

*7. (a) How are the figures in Problems 1, 2, and 3 alike?

(b) How are the figures different?

(c) Can you think of another way to draw a set of three lines in a plane? (Do not use the intersection of three lines in a point.)

(d) Copy and complete the following statement: "The intersections of three different lines in the same plane may consist of __?__, __?__, __?__, or __?__ points."

*8. If you look around you, you can find illustrations of sets of three lines like those in the figures you drew in these exercises. Now try to imagine three planes in space. Describe a situation which represents three planes in space having the following intersections.

(a) A point.

(b) Two parallel lines.

(c) Three parallel lines.

(d) The empty set.

10-3. Parallel Lines and Corresponding Angles

We have two lines r_1 and r_2, cut by the transversal t. Lines r_1 and r_2 do not intersect. In the language of sets we now have the following:

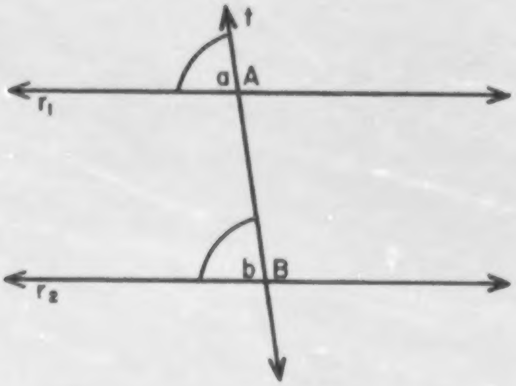


Figure 10-3a

$$r_1 \cap t \text{ is not the empty set,}$$
$$r_2 \cap t \text{ is not the empty set,}$$
$$r_1 \cap r_2 \text{ is the empty set.}$$

That is, r_1 and r_2 are parallel. Neither r_1 nor r_2 is parallel to t.

Class Exercise and Discussion

1. You are to make a drawing similar to Figure 10-3-a. Draw a line t, and locate points A and B about $1\frac{1}{2}$ inches apart. At point A, draw an angle with measure 70 for $\angle a$. At point B, use a measure of 40 for angle $\angle b$. Do the lines intersect if extended? If so, on which side of t, the left or the right?

2. Draw another figure making 30 the measure of $\angle a$ and 40 the measure of $\angle b$. Do the lines intersect if extended? If so, on which side of t, the left or the right?

3. Make at least six experiments of this kind with various measures for angles a and b. In at least two cases use the same measure for $\angle a$ that you use for $\angle b$. Record your results like this:

Measure of $\angle a$ in degrees	Measure of $\angle b$ in degrees	Intersection of r_1 and r_2
70	40	Left of t
30	40	right of t
40	40	?

4. Copy the following table. Predict whether r_1 and r_2 intersect and, if so, where. Make a drawing to check your prediction. (You may extend the table and choose your own measures of angles if you wish.)

Measure of $\angle a$ in degrees	Measure of $\angle b$ in degrees	Intersection of r_1 and r_2
50	80	
50	50	
50	40	

5. After class discussion of Problems 1 through 4, copy and complete the following statement:

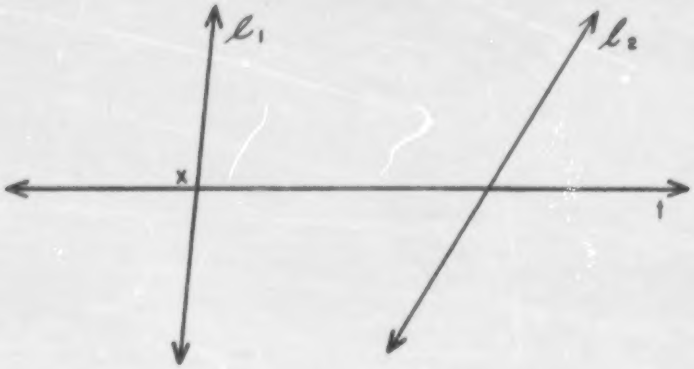
   Property 2. When, in the same plane, a transversal intersects two lines and a pair of corresponding angles have different ___?___, then the lines ___?___.

6. Consider the case where the corresponding angles are congruent (have equal measures). Write a statement for this case. Copy and complete:

   Property 2a. When, in the same plane, a transversal intersects two lines and a pair of corresponding angles are congruent, then the lines are ___?___.

Exercises 10-3

1. Make a figure like the one below. What angle forms with $\angle x$ a pair of corresponding angles? Label it $\angle p$. If angles x and p have the following measurements, do ℓ_1 and ℓ_2 intersect above t, below t, or are they parallel?



Copy and complete:

	Measure of x in degrees	Measure of p in degrees	ℓ_1 and ℓ_2		
			Are Parallel	Intersect above t	Intersect below t
(a)	120	140			
(b)	120	120			
(c)	120	90			
(d)	90	120			
(e)	90	90			
(f)	90	70			
(g)	40	90			
(h)	40	40			
(i)	40	20			

2. List several examples of parallel lines that you find in the classroom.

*3. Lay a yardstick or a ruler across some of the parallel lines you find in Problem 2 so as to form a transversal. Do this so that the intersections with the parallel lines are at a minimum distance apart. Measure this distance in each case. What is the measure of the angle formed by the transversal and each line when the minimum distance is found?

10-4. Converse (Turning a Statement Around)

We have seen that if certain things are true, then certain other things are true. For example:

(a) "If two angles are vertical angles, then the angles have the same measure."

Suppose we make a new statement by interchanging the "if" part and the "then" part. This is the new statement:

(b) "If two angles have the same measure, then the angles are vertical angles." A statement obtained by such an interchange is called a converse statement. In the example above, (b) is called the converse of (a). Since such an interchange in (b) brings us back to (a), we also call (a) the converse of (b).

If you "turn a true statement around," will the new statement always be true? Let us look at two statements and their converses and decide.

1. "If Mary and Sue are sisters, then Mary and Sue are girls."

Converse of 1: "If Mary and Sue are girls, then Mary and Sue are sisters." Is the original statement true? Is the converse also true?

Now consider the next statement:

2. "If Lief is the son of Eric, then Eric is the father of Lief."

Converse of 2: "If Eric is the father of Lief, then Lief is the son of Eric." Is the original statement true? Is the converse true?

We can see from these two illustrations that, if a statement is true, a converse obtained by interchanging the "if" part and the "then" part, may be true or may be false.

3. Is statement (a) above, dealing with vertical angles, true? Is the converse statement, (b), true? We cannot accept a converse statement as always being true. Sometimes when a true statement is "turned around," a converse is true. Sometimes when a true statement is "turned around," a converse is false.

Exercises 10-4

1. Make a drawing for which a converse of statement (a) in Section 10-4 is not true. Must any two angles which have the same measure always be vertical angles?

2. For each of the following statements write "true" if the statement is always true; "false" if the statement is sometimes false.

   (a) If Blackie is a dog, then Blackie is a cocker spaniel.

   (b) If it is July 4th, then it is a holiday in the United States.

   (c) If Robert is the tallest boy in his school, Robert is the tallest boy in his class.

   (d) If an animal is a horse, the animal has four legs.

   (e) If an animal is a bear, the animal has thick fur.

   (f) If Susan is Mark's sister, then Mark is Susan's brother.

3. Write a converse for each statement in Problem 2 and tell whether the converse is true or false.

4. Read the following statements. Write "true" if the statement is always true; "false" if the statement is sometimes false.

   (a) If a figure is a circle, then the figure is a simple, closed curve.

   (b) If a figure is a simple closed curve composed of three line segments, then the figure is a triangle.

   (c) If two angles are congruent, they are right angles.

   (d) If two lines are parallel, then the lines have no point in common.

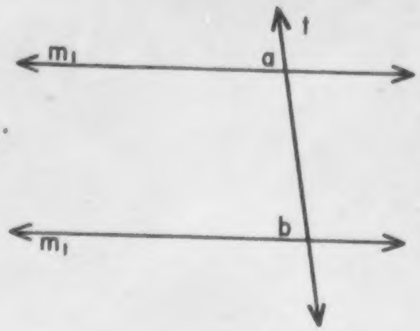
   (e) If two angles are supplementary, they are adjacent.

   (f) If two adjacent angles are both right angles, they are supplementary.

5. (a) Write a converse statement for the following property.

Property 2a: If, in the same plane, a transversal intersects two lines, and a pair of corresponding angles are congruent, then the lines are parallel.

(b) To test the possible truth of the converse, draw m_1 and m_2, and transversal t as in the figure. Are corresponding angles congruent? Now draw any other transversal of m_1 and m_2. Call it t_1. Measure the angles in each pair of corresponding angles along t_1. Are they congruent?



Compare your results with those of your classmates.

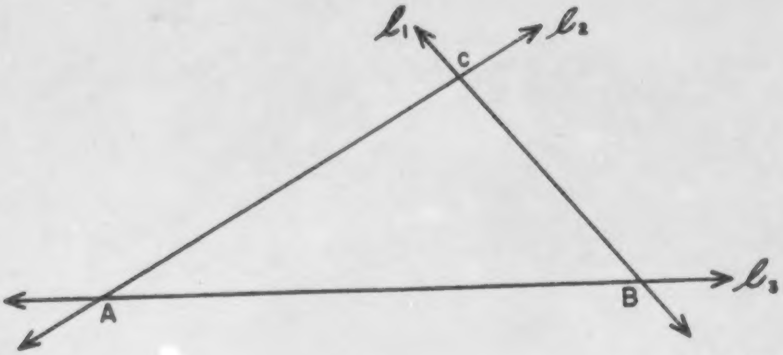
(c) On the basis of this work, do you think the converse of Property 2a stated in (a) above is true or false?

*6. Write a converse statement for Property 2: If, in the same plane, a transversal intersects two lines, and a pair of corresponding angles have different measures, then the lines intersect. Does the statement seem to be true or false? You can test your results by making drawings as in Problem 5.

*7. Can a converse for a false statement be true? If so, can you give an example?

10-5. Triangles

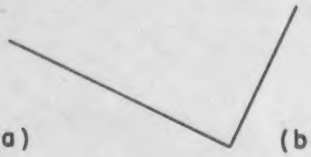
You have been discovering angle relations in a figure composed of three lines, two parallel lines and a transversal. Suppose the lines are arranged so that each line is a transversal for the other two lines. Would such a figure appear like this figure?



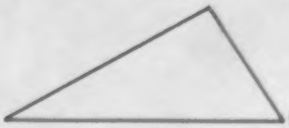
In the drawing above, ℓ_1 is a transversal for ℓ_2 and ℓ_3; ℓ_2 is a transversal for ℓ_1 and ℓ_3; ℓ_3 is a transversal for ℓ_1 and ℓ_2. A, B, and C are three points and $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$ are segments joining them in pairs. In Chapter 4 (Non-Metric Geometry) the union of three points not on the same line and the segments joining them in pairs was called a triangle. Our sketch, according to this definition, contains triangle ABC. The points A, B, and C are called the vertices of the triangle and the segments $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$ are called the sides of the triangle.

Which of the figures below are triangles? If a figure is not a triangle, tell which requirement of the definition is lacking.

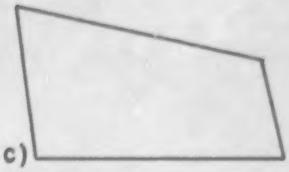
(a)



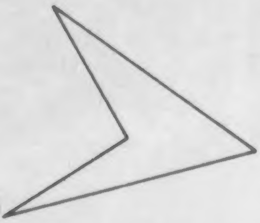
(b)



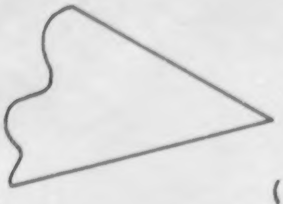
(c)



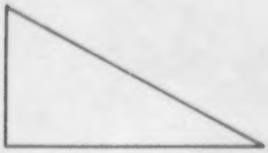
(d)



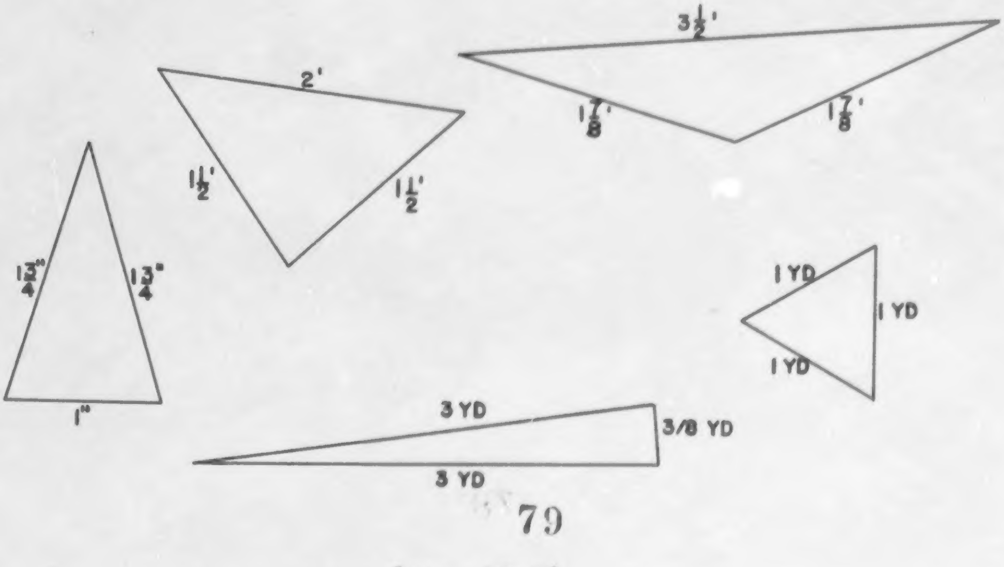
(e)



(f)

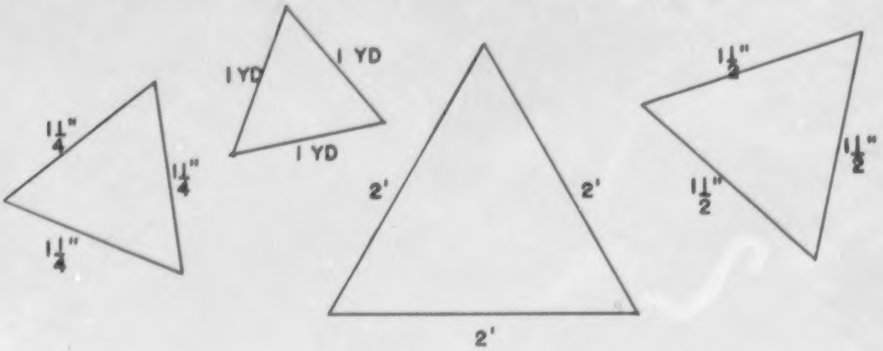


The triangles in the group below are members of a particular set of triangles because they have a common property. What do you notice about the lengths of the sides of the triangles? (The symbol 3" means three inches. The symbol 3' means three feet.)



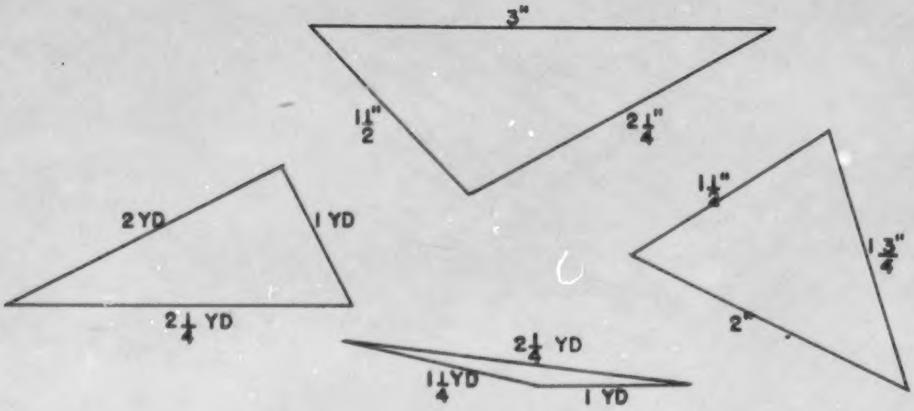
Note that each triangle has at least two sides which are equal in length. The triangles of this set, and all other triangles that have this property, are called isosceles triangles.

The triangles in the group below are members of another particular set of triangles. What is the common property in this set of triangles?



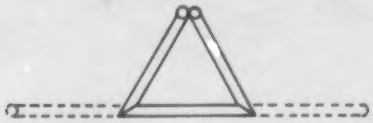
Note that each triangle has three sides which are equal in length. Another way of saying this is that all sides in the same triangle have equal measures. All triangles of this set, and all other triangles that have this property, are called equilateral triangles. The word "equilateral" comes from the words, "equi" meaning equal and "lateral" meaning side.

The triangles in the following group are members of still another set of triangles. Do the triangles in this set appear to have a property in common?

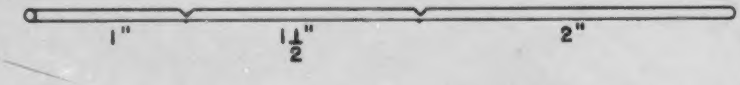

3"
$1\frac{1}{2}$"
$2\frac{1}{4}$"
2 YD
1 YD
$2\frac{1}{4}$ YD
$1\frac{1}{4}$"
$1\frac{3}{4}$"
2"
$2\frac{1}{4}$ YD
$1\frac{1}{4}$ YD
1 YD

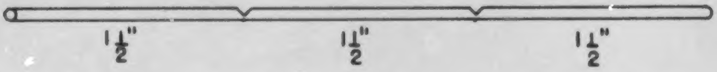
Do two sides in any of the above triangles have the same measure? Do any of the triangles have three sides with equal measures? No two sides in any of the triangles above have the same measure. The triangles in this particular set, and all other triangles having the same property, are called scalene triangles.

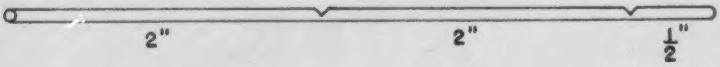
Here is a picture of a soda straw and here it is creased at two points ready to be folded so that the ends come together like this:-

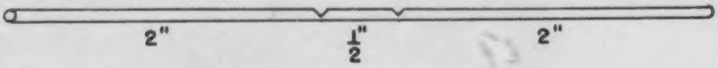


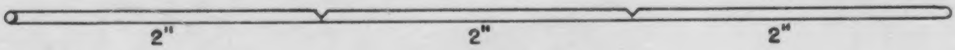
What kinds of triangles could be represented in this way by folding the following straws:



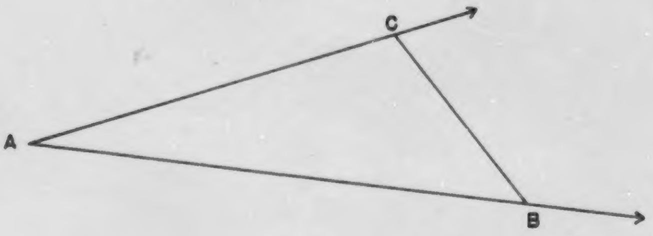








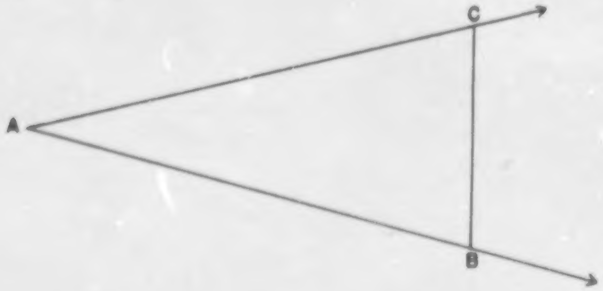
Now you are to draw some triangles. Draw an angle, and name the vertex A. Locate a point B on one ray of the angle, and a point C on the other ray. Draw segment $\overline{BC}$ Your drawing should look something like this.



The union of $\overline{AB}$, $\overline{BC}$, and $\overline{AC}$ is called "triangle ABC." Notice that in your drawing, points C and B are the only points shared by angle A and side $\overline{BC}$. Angle A and side $\overline{BC}$ are said to be opposite each other because their intersection consists only of the endpoints of $\overline{BC}$. Is $\angle B$ opposite $\overline{AC}$? Why? Is $\angle C$ opposite $\overline{AC}$? Why? What other angle and side are opposite to each other in your drawing? Why?

[sec. 10-5]

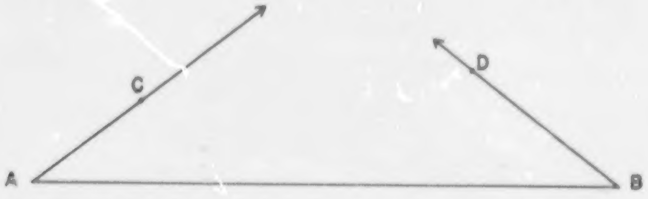
Draw another triangle as before, but this time very carefully locate point B and point C so that $\overline{AB}$ and $\overline{AC}$ are congruent as shown below:



What kind of a triangle is Δ ABC? Why? Make a copy of angle B and try to fit the copy upon angle C. What does this procedure suggest about the measure of angles B and C? Make several more copies of the isosceles triangles on the second page of this section. What do you observe? Copy and complete the following statement:

Property 3. If two sides of a triangle are ___?___ in length, then the angles ______?______ these sides have ___?___ measures.

Now make a drawing like the one below. Draw AB about four inches in length. Draw 40 degree angles at A and B as shown. Locate points C and D at some convenient place on the rays of these angles.



Notice that we have drawn two angles whose measures are the same. Extend $\overline{AC}$ and $\overline{BD}$. Name their point of intersection E. Thus triangle ABE is formed.

What do you observe about the sides opposite $\angle A$ and $\angle B$ in this triangle? Make several other sketches according to the directions given. Keep angles A and B acute and equal in measure, but vary the measure for different triangles. What do you observe about the sides in each figure? To what special set of triangles do your figures appear to belong?

Copy and complete the following statement:

If two angles of a triangle are _____?_____ in measure, then the sides _____?_____ these angles are _____?_____ in length.

Have you noticed that this statement resembles the statement completed as Property 3 previously? What new word learned by studying Section 10-4 could be used to describe the relationship between these two statements? We will call this statement the converse of Property 3.

Class Discussion Problems

1. Could the three sets of triangles we have discussed (isosceles, equilateral, and scalene) be determined by some common property pertaining to angle measures instead of side measures? What advantages or disadvantages would this method of classification have?
2. How could the converse of Property 3 be used to show that a triangle having all three of its angles equal in measure must necessarily be an equilateral triangle?

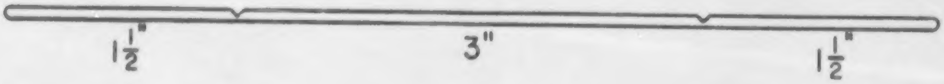
Exercises 10-5

1. Draw an isosceles triangle.
2. Draw an equilateral triangle.
3. Draw a scalene triangle.
4. If a triangle is isosceles is it also equilateral? Explain your answer.

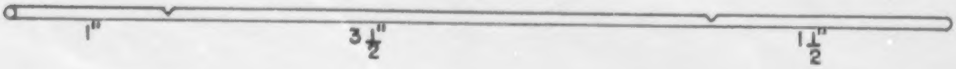
5. If a triangle is equilateral, is it also isosceles? Explain your answer.

6. Draw an equilateral triangle and letter the vertices P, Q, and R. Match each angle with its opposite side by listing them in pairs.

7. (a) Could a triangle be represented by folding the soda straw shown in this figure? Explain your answer.

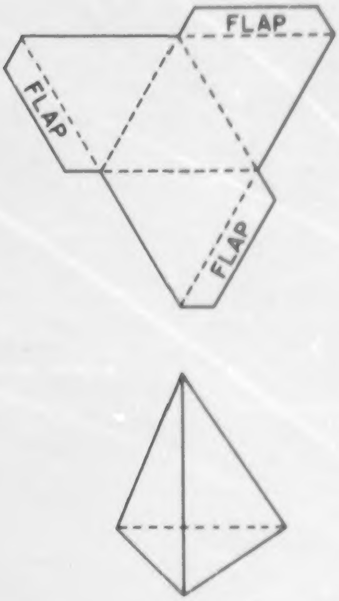


(b) Could a triangle be represented by folding the soda straw shown here? Explain your answer.



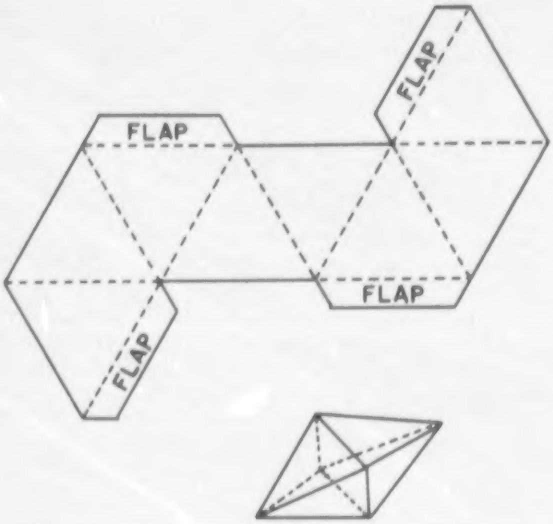
(c) State a property about the lengths of the sides of a triangle as suggested by your observations in parts (a) and (b).

8. OPTIONAL: Copy the following patterns. Fold on the dotted lines after cutting along the edges, and paste the flaps. Note the equilateral triangles. Try to enlarge these patterns for better results.



tetrahedron

(four faces)



octahedron

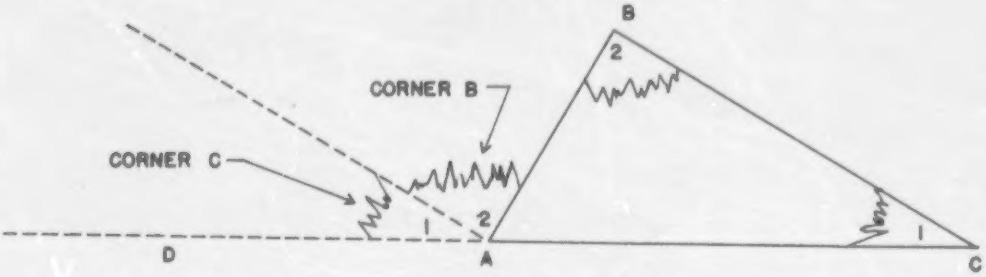
(eight faces)

10-6. Angles of a Triangle

In the previous section we studied special properties for certain triangles. In this section we come to a property which is true for all triangles.

Class Exercises and Discussion

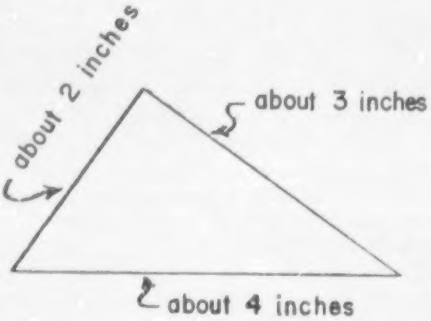
1. Draw a triangle, making each side about two or three inches in length. Cut out the triangular region by cutting along the sides of the triangle. Tear off two of the corners of this region and mount the whole figure on cardboard or a sheet of paper as shown in the figure below. Note: Corners B and C are placed around the vertex A. (The corners may be pasted or stapled in place.)



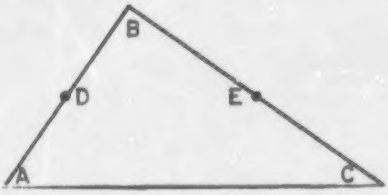
(a) Find the measures of the three angles with vertices at A. Find the sum of these three measures and compare your results with those of your classmates. In the drawing above, does it appear that $\overrightarrow{AD}$ and $\overrightarrow{AC}$ are on the same line? Does this appear to be true on the figure you made?

(b) What do you observe about angle 1, angle 2, and angle BAC in this new arrangement?

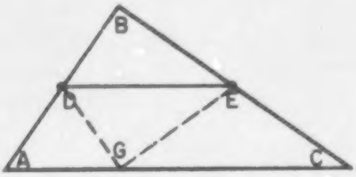
2. (a) Draw a triangle making the longest side about 4 inches long, one of the remaining two sides about 3 inches long, and the third side about 2 inches long. Cut out the interior of the triangle.



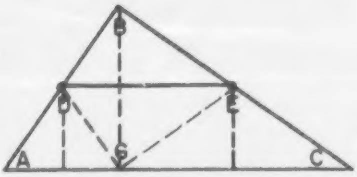
(b) Label the vertices A, B, and C in the interior as shown in the drawing at the right. Mark off the midpoint of $\overline{AB}$. Label the midpoint D. (The midpoint is halfway from one endpoint of a line segment to the other endpoint.) Find the midpoint of $\overline{BC}$. Label the midpoint E. $\overline{AD}$ and $\overline{DB}$ will have the same length. $\overline{BE}$ and $\overline{EC}$ will have the same length.

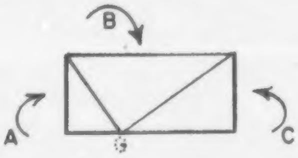


(c) Draw a line segment joining D and E. Fold downward the portion of the triangle containing the vertex B along the line segment $\overline{DE}$ so that the vertex B falls on $\overline{AC}$. Label the point where B falls on $\overline{AC}$ as G. The fold is along the segment $\overline{DE}$.



(d) Fold the portion containing A to the right so that the vertex A falls on the point G. Fold the portion containing C to the left so that the vertex C also falls on point G. The resulting figure will be a rectangle.



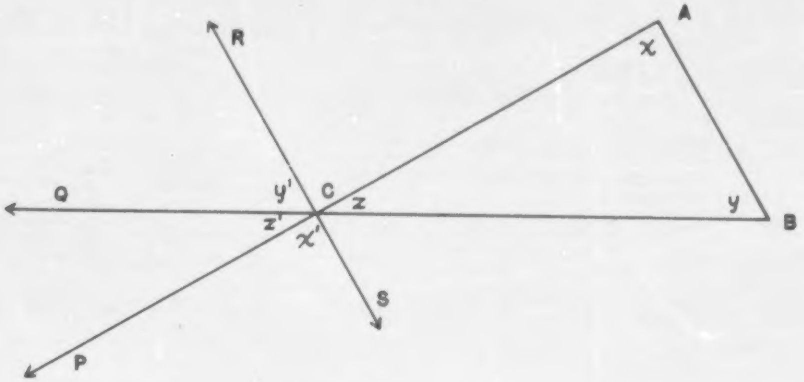


(e) What appears to be true about the sum of the measures of the angles A, B, and C.

(f) Does this experiment work with other triangles? Check your results with those of your classmates.

(g) Does the property you found in Problem 1 agree with these observations?

3. Consider the triangle ABC and the rays $\overrightarrow{AP}$ and $\overrightarrow{BA}$ shown below. Line RS is drawn through point C so that the measure of $\angle y$ and the measure of $\angle y'$ are equal Here we are using a new notation, y'. y' is read "y prime." (In this problem we are using this notation in naming angles.)



Use a property to explain "why" for each of the following whenever you can:

(a) Is $\overleftrightarrow{RS}$ parallel to $\overline{AB}$? Why?

(b) What kind of angles are the pair of angles marked x and x'? Is $m(\angle x) = m(\angle x')$? Why?

(c) What kind of angles are the pair of angles marked z and z'? Is $m(\angle z) = m(\angle z')$? Why?

(d) $m(\angle y) = m(\angle y')$ Why?

(e) $m(\angle x) + m(\angle y) + m(\angle z) = m(\angle x') + m(\angle y') + m(\angle z')$ Why?

(f) $m(\angle x) + m(\angle y) + m(\angle z)$ is the sum of the measures of the angles of the triangle. Why?

(g) $m(\angle x') + m(\angle y') + m(\angle z') = 180$ Why?

(h) $m(\angle x) + m(\angle y) + m(\angle z) = 180$ Why?

(i) We conclude therefore that the sum of the measures of the angles of the triangle is 180. Why?

This is a proof of Property 4.

Property 4. The sum of the measures in degrees of the angles of any triangle is 180.

We will not study many proofs this year, but in some exercises you may be asked to try to discover a proof. As you study more geometry in later years you should become quite able to discover proofs.

Notice that in this proof we drew just any triangle. Does the proof apply to all triangles? If you are in doubt about this, you might draw some other triangles quite different in shape from the one in this section, label points, angles, segments, rays, and lines in the same way. Then, try the proof above for the figure you have drawn.

Exercises 10-6

1. What is the measure of each angle of an equilateral triangle?

2. What is the measure of the third angle of the triangles if two of the angles of the triangle have the following measures?

   (a) 40 and 80.

   (b) 100 and 50.

   (c) 70 and 105.

   (d) 80 and 80.

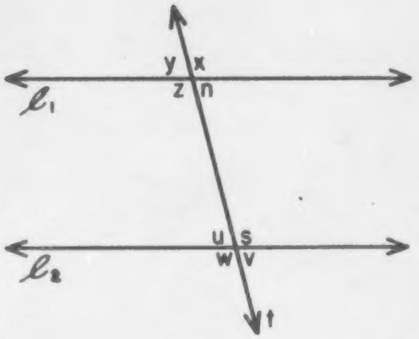
3. Suppose one angle of an isosceles triangle has a measure of 50. Find the measures of the other two angles. Are two different sets of answers possible?

4. If two triangles, ABC and DEF, are drawn so that $m(\angle BAC) = m(\angle EDF) = m(\angle BCA) = m(\angle EFD)$, what will be true about angles ABC and DEF? Upon what property is your answer based?

5. In each of the following the measures of certain parts of the triangle ABC are given, those of the sides in inches and those of the angles in degrees. You are asked to find the measure of some other part. In each case, give your reason.

Given	Find	Property
(a) $m(\angle ABC) = 60$, $m(\angle BCA) = 40$.	$m(\angle CAB)$	?
(b) $m(\angle CAB) = 52$, $m(\angle BCA) = 37$.	$m(\angle ABC)$	?
(c) $m(\angle ABC) = 40$, $m(\overline{AB}) = 2$, and $m(\overline{AC}) = 2$.	$m(\angle ACB)$	?
(d) $m(\overline{AB}) = 3$, $m(\overline{AC}) = 3$, $m(\overline{BC}) = 3$.	$m(\angle BCA)$	?
(e) $m(\angle BAC) = 100$, $m(\angle BCA) = 40$, and $m(\overline{AB}) = 4$.	$m(\overline{AC})$	?

*6. In the figure at the right ℓ_1 and ℓ_2 are parallel.



(a) Does $m(\angle y) = m(\angle n)$? Why?

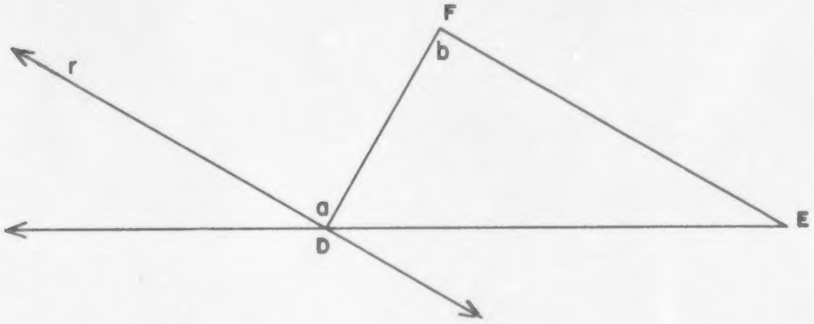
(b) Does $m(\angle y) = m(\angle u)$? Why?

(c) Try to prove that $m(\angle n) = m(\angle u)$.

(d) Discuss with your classmates the things which make a good proof.

(e) After the class discussion, rewrite your proof in accordance with the points brought out by you and your classmates.

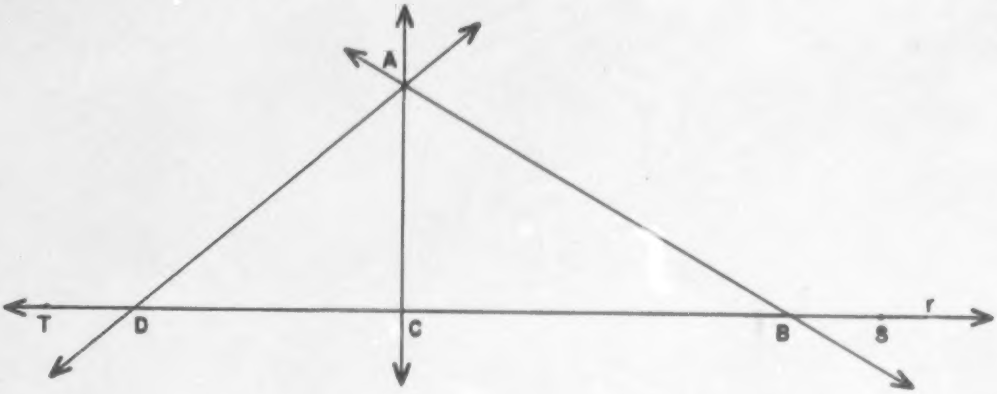
*7. In problem 6, it has been shown that the measures of angles a and b in the figure below are equal if line r is parallel to $\overleftrightarrow{EF}$. Use this property to prove that the sum of the measures of the angles of triangle DEF shown below is 180.



10-7. Parallelograms

Distance Between Parallel Lines

Look at the figure below showing lines through point A and intersecting line **r**.



Use a protractor to check the measures of the angles given below.

m(∠ SDA) = 40 m(∠ SCA) = 90 m(∠ SBA) = 150

Give the following measures.

m(∠ TDA) = ? m(∠ TCA) = ? m(∠ TBA) = ?

Measure the segments $\overline{AD}$, $\overline{AC}$, and $\overline{AB}$.
Which is the shortest?
Copy the figure and draw other lines through A which intersect r.
Measure the segments on these lines from A to the intersections of the lines with r.
Do you find any of these segments that are shorter than $\overline{AC}$?
Note that $\overleftrightarrow{AC}$ is perpendicular to r.
On the basis of your experience here, copy and complete the following statement:

> The shortest segment from a point A to a line r is the segment which is _________ to r.

The length of this shortest segment is often called the distance from A to r.

In the following figure, k_1 and k_2 represent parallel lines. Lines a, b, and c are perpendicular to K_2 and go through three points: D, E, and F of K_1. That is, the lengths of $\overline{FA}$, $\overline{EB}$, and $\overline{DC}$ are the distances from F, E, and D to line k_2. One often draws a small square, as in the figure at A, B, C, to indicate that an angle is intended to be a right angle.

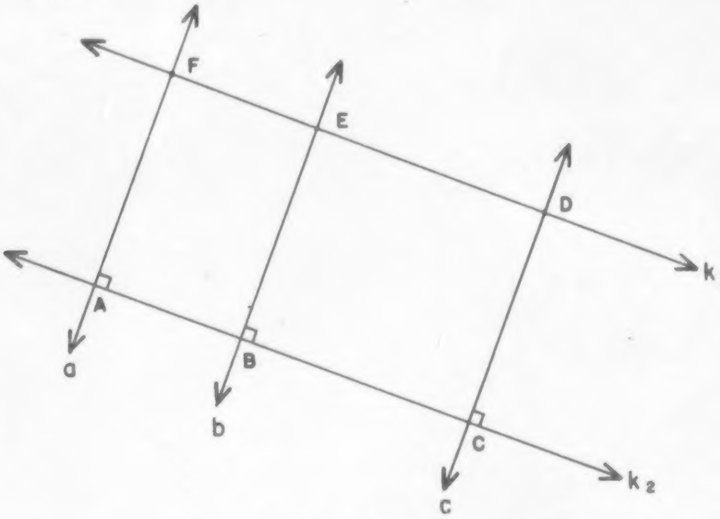
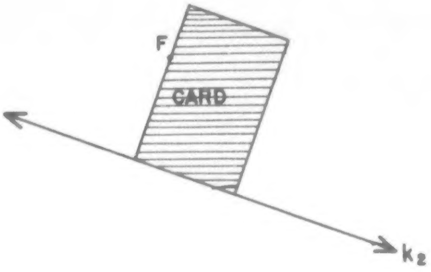


Figure 10-7-a

Drawing a line, such as C through D and perpendicular to k_2, can be conveniently done by using a card or sheet of paper with a square corner. The figure at the right illustrates this.

In Figure 10-7-a above there are 21 other right angles besides those marked. Can you find them? How do you know they are all right angles?

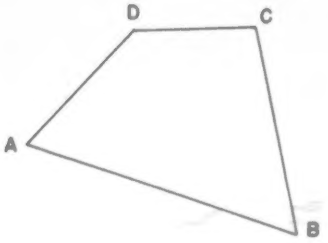


In Figure 10-7-a measure the lengths of $\overline{FA}$, $\overline{EB}$, and $\overline{DC}$. Do they appear to be equal in length? This common length is called the distance between lines k_1 and k_2. Thus the *distance between two parallel lines* may be described as the length of any segment contained in a line perpendicular to the two lines, and having an endpoint on each of the lines.

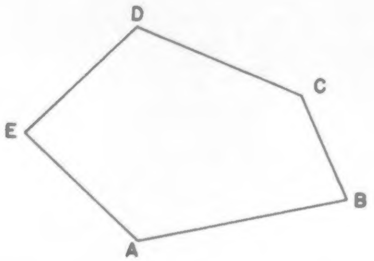
Parallelograms

In Chapter 4 you were introduced to the idea of a simple closed curve. Any simple closed curve which is a union of segments may be called a *polygon*. In later work you may sometimes see the word "polygon" applied to curves which are not simple, but any polygon we study in this chapter will be a simple closed curve. Unless it is indicated otherwise, we shall also understand that a polygon lies in a plane.

Polygons with different numbers of sides (i.e., segments) are given special names. You already know that a polygon with three sides is called a triangle. Similarly a polygon with *four* sides is called a *quadrilateral*, and a polygon with *five* sides is called a pentagon.



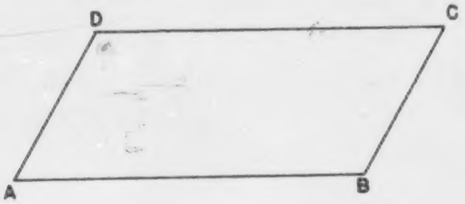
Quadrilateral



Pentagon

In a quadrilateral two sides (segments) which do not intersect are called opposite sides. (How would you describe two opposite sides by referring to their intersection?) Name the pairs of opposite sides in the above quadrilateral.

A particularly important kind of quadrilateral is the parallelogram. This is a quadrilateral whose opposite sides lie on parallel lines. The figure ABCD below represents a parallelogram. Name its pairs of opposite sides.

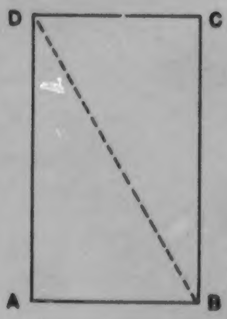


In the future, if two segments lie on parallel lines we will speak of the segments as parallel. Thus we can say that the opposite sides of a parallelogram are parallel.

Property 5. Opposite sides of a parallelogram are parallel and congruent.

We already know opposite sides are parallel. To test for congruence, measure the sides in the parallelogram above. Do you find that the opposite sides have equal length? Draw several parallelograms and measure the lengths of the pairs of opposite sides. Do you agree with the property (underlined) above?

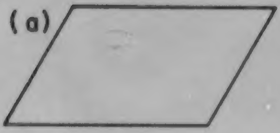
The figure at the right represents a rectangular sheet of paper. (Notice that a rectangle is a special kind of parallelogram.) Take such a piece of paper and tear or cut it into two parts by first folding it along the dotted line as shown. By laying one piece on the other, show that triangle ABD and BCD have the same size. Is the area of one of the triangles equal to one half the area of the rectangle? Why?

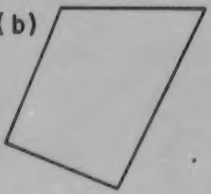


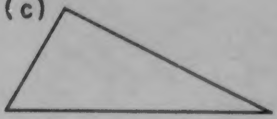
In Problem 5 below you are asked to repeat the above steps using parallelograms of different shapes.

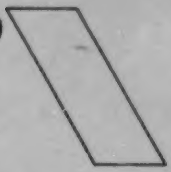
Exercises 10-7

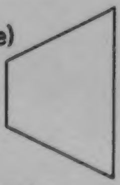
1. Identify several pairs of parallel lines in your classroom, and measure the distances between them.
2. Identify several examples of parallelograms in your classroom.
3. Which of the following figures are parallelograms, assuming that segments which appear to be parallel are parallel?

(a)


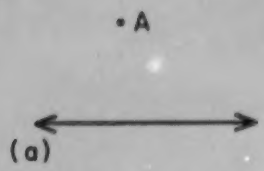
(b)


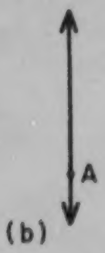
(c)


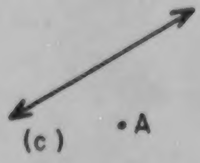
(d)


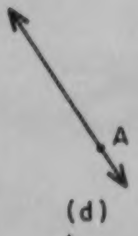
(e)


4. Draw perpendiculars to lines in approximately the positions illustrated through points A as indicated. Use a separate piece of paper. Do not draw in your book.



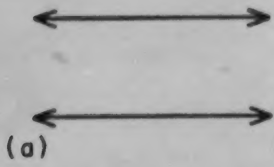


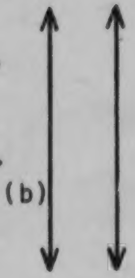


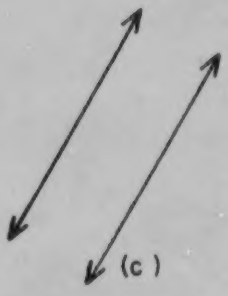


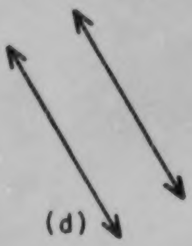
5. Draw a parallelogram and cut carefully along its sides. The resulting paper represents the interior of the parallelogram. Draw a diagonal (a line joining opposite vertices) and cut the paper along this diagonal. Compare the two triangular pieces. What do you conclude about these triangular pieces? Carry out the same process for two other parallelograms of different shapes. Write a statement that appears to be true on the basis of your experience in the problem.

6. For each of the sets of parallel lines in the figure below, draw a line perpendicular to one of them. (Do not write in your book. Copy the lines in approximately these positions on a separate piece of paper.) Are the lines which are perpendicular to one of two parallel lines perpendicular to the other also?

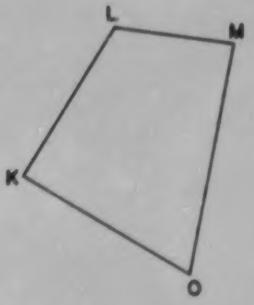








7. The following questions refer to a figure which is a quadrilateral, as suggested by the drawing. Each question, however, involves a different quadrilateral.



(a) $\overline{KL}$ is parallel to $\overline{OM}$, $\overline{LM}$ is parallel to $\overline{KO}$, $\overline{KL}$ has a length of 3 in. and $\overline{OK}$ a length of 6 in. What are the lengths of $\overline{LM}$ and $\overline{OM}$?

(b) $\overleftrightarrow{KL} \cap \overleftrightarrow{OM}$ is the empty set, $\overleftrightarrow{LM} \cap \overleftrightarrow{OK}$ is the empty set. $\overline{LM}$ has a length of 4 in. and $\overline{OM}$ is three times as long as $\overline{LM}$. Find the lengths of $\overline{KL}$ and $\overline{OK}$.

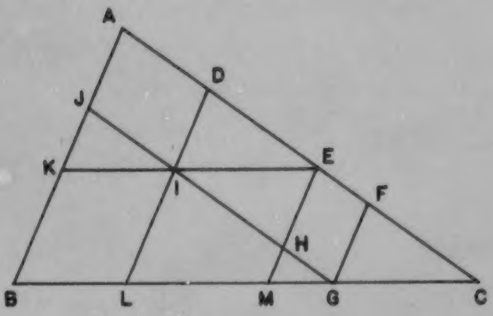
(c) $\overleftrightarrow{LM} \cap \overleftrightarrow{OK}$ is the empty set, $\overleftrightarrow{OM} \cap \overleftrightarrow{KL}$ is not the empty set. Can two opposite sides have the same length? Can both pairs of opposite sides have this property? (Draw figures to illustrate this.)

*8. (a) Copy and complete the following statement of a property:

If, in a plane, a line is perpendicular to one of several parallel lines, then it is ________________.

(b) Prove the property. (You may assume that a line perpendicular to one of two parallel lines does intersect the other.)

*9. In triangle ABC shown at the right, assume that the segments $\overline{AB}$, $\overline{DL}$, $\overline{EM}$, $\overline{FG}$ are parallel; assume that the segments $\overline{AC}$ and $\overline{JG}$ are parallel; and, assume that the segments $\overline{BC}$ and $\overline{KE}$ are parallel.



(a) List the parallelograms in the drawing. (There should be 10 parallelograms.)

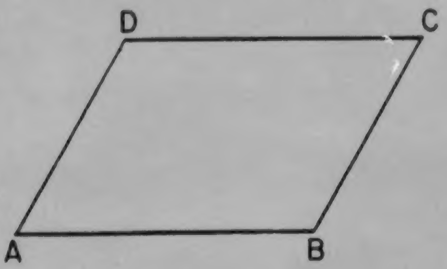
(b) Without measuring, list the segments in the above figure which are congruent to $\overline{AJ}$.

(c) Without measuring, list the segments in the above figure which are congruent to $\overline{BK}$.

(d) Without measuring, list the segments in the above figure which are congruent to $\overline{GI}$.

10-8. Areas of Parallelograms and Triangles

A quadrilateral is a four-sided figure. Any quadrilateral, in which the opposite sides lie on parallel lines, is called a parallelogram. What have you already discovered about the lengths of the opposite sides of a parallelogram? Figure ABCD, shown at the right, is a parallelogram. Which pairs of sides in this figure are congruent?



At each vertex of a parallelogram there is an angle whose interior contains the interior of the parallelogram. At the vertex A in the figure on the previous page, the interior of angle BAD contains the interior of the parallelogram. What are the names of the other angles which contain the interior of the parallelogram? How many such angles are there altogether? These are called the angles of the parallelogram.

If the sides of a parallelogram are extended by dotted lines, as shown in the Figure 10-8-a,

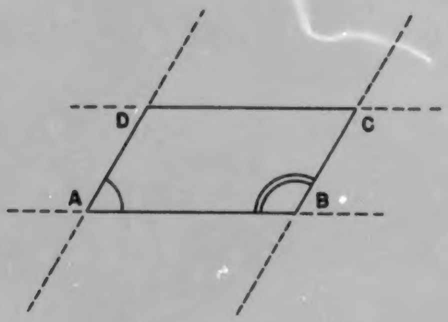


Figure 10-8-a

there are several pairs of parallel lines cut by a transversal. How many transversals do you find?

Copy Figure 10-8-a. Mark the angle of the parallelogram at vertex A as shown. We will call this angle of the parallelogram "angle A." Similarly, mark angle B of the parallelogram as shown. Mark all the angles in your figure which are congruent to angle A in the same way that angle A is marked. Do the same for angles which are congruent to angle B. In each case give the reason why the label is correct.

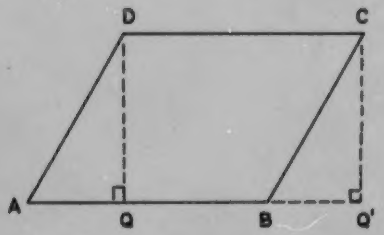
If you have proceeded correctly, all the angles determined by the lines in the figure are now labeled. On the basis of the results just obtained, complete the following statements:

$m(\angle A) + m(\angle B) = \underline{\quad ? \quad}$

Property 6a. The angles of a parallelogram at two consecutive vertices are ? .

Property 6b. The angles of a parallelogram at two opposite vertices are ? .

According to the results above, what can you conclude about the parallelogram if $\angle A$ is a right angle? Can you be sure it is a rectangle? (Do you recall from Chapter 8 how to find the area of a rectangle?) If the figure is not a rectangle then $\angle A$ and $\angle B$ are not right angles, and one of them is an acute angle. Why? Suppose the acute angle is $\angle A$. From point D, draw the segment $\overline{DQ}$ perpendicular to the base $\overline{AB}$ of the parallelogram as shown in the figure at the right. Since we know that $\overline{AD}$ and $\overline{BC}$ are congruent, imagine the triangle AQD moved rigidly, that is, without changing its size and shape, into the position of triangle BQ'C. Then point Q' lies on the extension of $\overline{AB}$. How do you know?

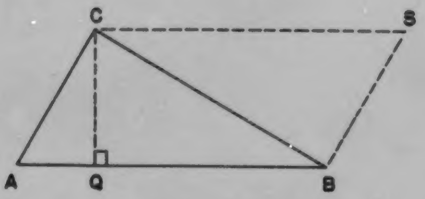


The figure QQ'CD is therefore a rectangle. (How do you know it has right angles at C and D?) Moreover the rectangle QQ'CD and the parallelogram ABCD are made up of pieces of the same size and thus have the same area according to the Matching Property in Section 2 of Chapter 7. To find the area of the parallelogram it is only necessary therefore to find the area of the rectangle, which we already know how to do. If you do not recall this, refer to Chapter 8.

Notice that the base $\overline{AB}$ of the parallelogram is congruent to the side $\overline{QQ'}$ of the rectangle. How do you know that is true? The other side of the rectangle, $\overline{QD}$, is a segment perpendicular to the parallel lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$. Such a line is called an altitude of the parallelogram to the base $\overline{AB}$. The length of the altitude is the distance between the two parallel lines and, as we have already seen, is the same wherever measured. Thus it actually makes no difference whether we consider the altitude from D or that from C or that from any point of $\overline{DC}$. Also, either side of a parallelogram may be considered as a base.

On the basis of the discussion above copy and complete the following statement: "The number of square units of area in the parallelogram is the ___?___ of the number of linear units in the ______?______ and the number of linear units in the ___?___ to this base.

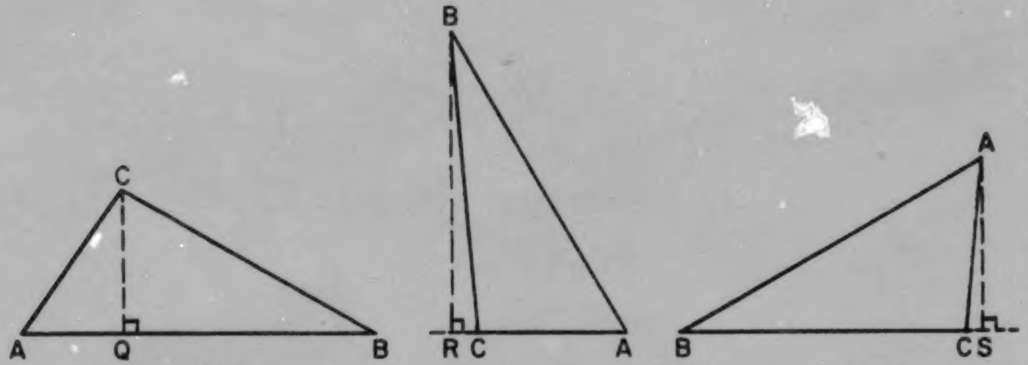
Notice that if a parallelogram is a rectangle, the lengths of the base and the altitude are simply the "length" and "width" of the rectangle, so the method above is the same as our earlier one in Chapter 8.



Consider any triangle ABC as shown at the right. Through C and B draw lines parallel to segments $\overline{AB}$ and $\overline{AC}$ and meeting in some point S. The figure ABSC is therefore a parallelogram. The segment $\overline{CQ}$ through C perpendicular to line $\overleftrightarrow{AB}$ is called the altitude of the triangle ABC to the base $\overline{AB}$. The length of altitude $\overline{CQ}$ is the distance from C to line $\overleftrightarrow{AB}$. Notice that $\overline{AB}$ and $\overline{CQ}$ are also a base and an altitude of the parallelogram.

In Section 7 you discovered that the areas of triangles ABC and SCB are the same. Since the two triangular regions cover the whole parallelogram and its interior, it follows that the area of the triangle ABC is one half that of the area of the parallelogram ABSC. Using the method above for the parallelogram, copy and complete the following statement: "The number of square units in the area of a triangle is _______ the _______ of the number of linear units in the _______ and the number of linear units in the _______ to this base."

Since any side of a triangle may be considered as the base, this rule actually gives three ways of finding the area of any triangle. This is illustrated in the following figures showing the same triangle using each of the 3 sides as the base:



BASE $\overline{AB}$ ALTITUDE $\overline{QC}$

BASE $\overline{AC}$ ALTITUDE $\overline{RB}$

BASE $\overline{CB}$ ALTITUDE $\overline{SA}$

Figure 10-8-b

In the previous illustration, the same triangle has been shown in three positions, so that in each case the base is shown as horizontal. This is not necessary, and you will want to practice thinking of the bases and corresponding altitudes in different positions.

For example, it might have been better not to move the triangle but to show the three cases as follows:

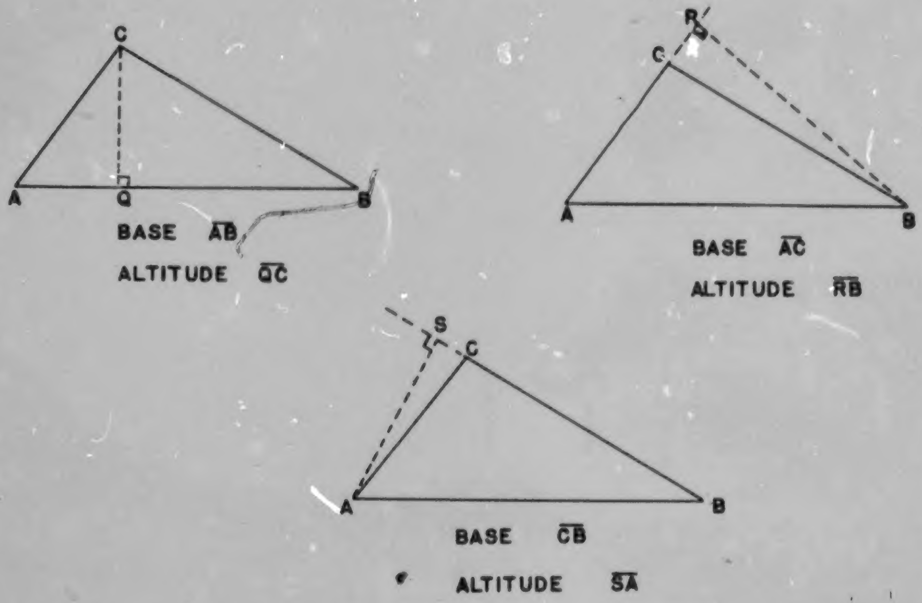
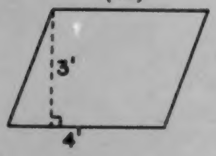

C
A
Q
B
BASE $\overline{AB}$
ALTITUDE $\overline{QC}$
R
C
A
B
BASE $\overline{AC}$
ALTITUDE $\overline{RB}$
S
C
A
B
BASE $\overline{CB}$
ALTITUDE $\overline{SA}$

Figure 10-8-c

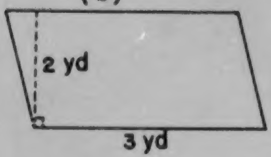
Exercises 10-8

1. Find the areas of the parallelograms shown, using the dimensions given.

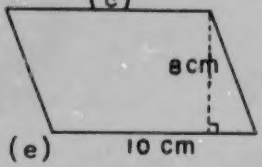
(a)



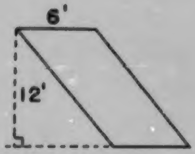
(b)



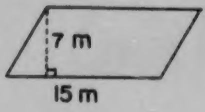
(c)



(d)

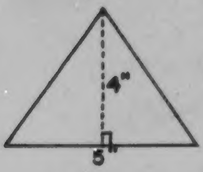


(e)

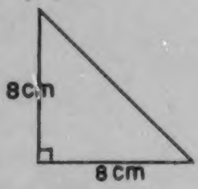


2. Find the areas of the triangles shown using the dimensions given.

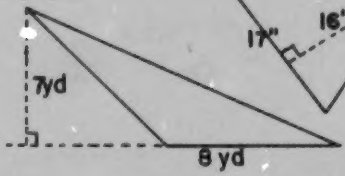
(a)



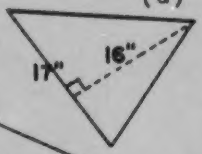
(b)



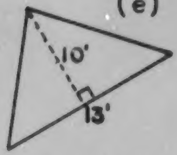
(c)



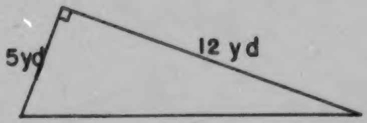
(d)



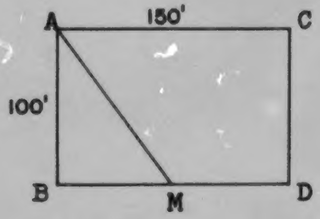
(e)



3. A right triangle has sides of 5 yards and 12 yards as shown. Find the number of square yards in its area.



4. A man owned a rectangular lot 150 ft. by 100 ft. From one corner, A, a fence is placed to the point M in the center of the longer opposite side as shown.



(a) Find the area of ABDC.

(b) Find the area of AMB.

(c) Find the area of AMDC.

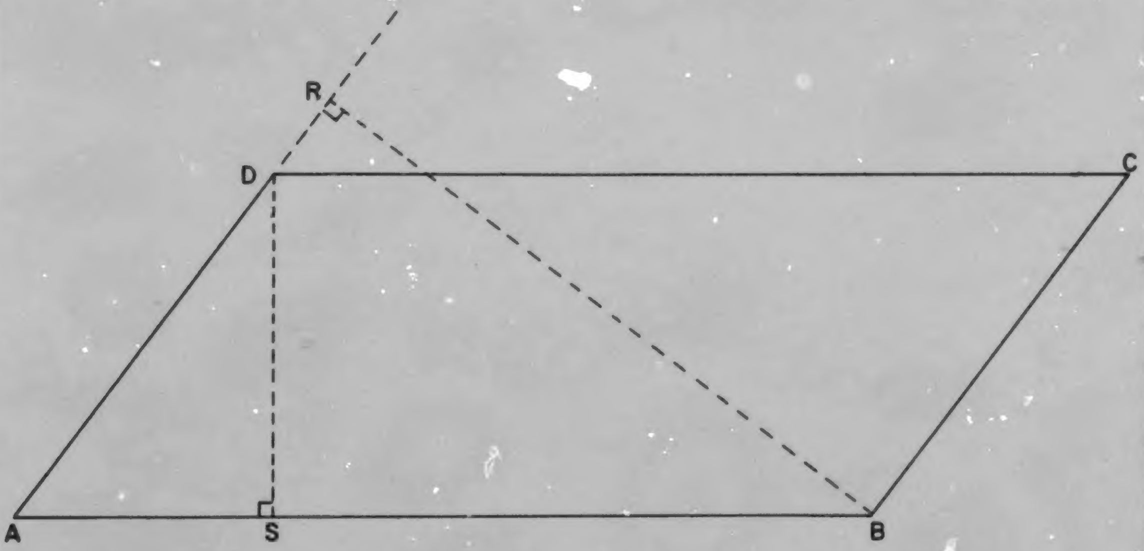
5. If b is the number of linear units in the length of the base of a parallelogram and h the number of linear units in the length of the altitude to this base, write a number sentence showing how to find the number A of square units in the area of the parallelogram.

6. If b is the number of linear units in the length of the base of a triangle and h the number of linear units in the length of the altitude to the base, write a number sentence showing how to find the number A of square units in its area.

7. (a) In the drawing below, measure $\overline{AB}$ and $\overline{DS}$. Using these measures, find the area of the parallelogram.

   (b) Measure $\overline{AD}$ and $\overline{RB}$. Using these measures, find the area of the parallelogram.

   (c) Do your results in (a) and (b) agree? Since measurement is approximate, they may not be exactly the same, but they should be close.



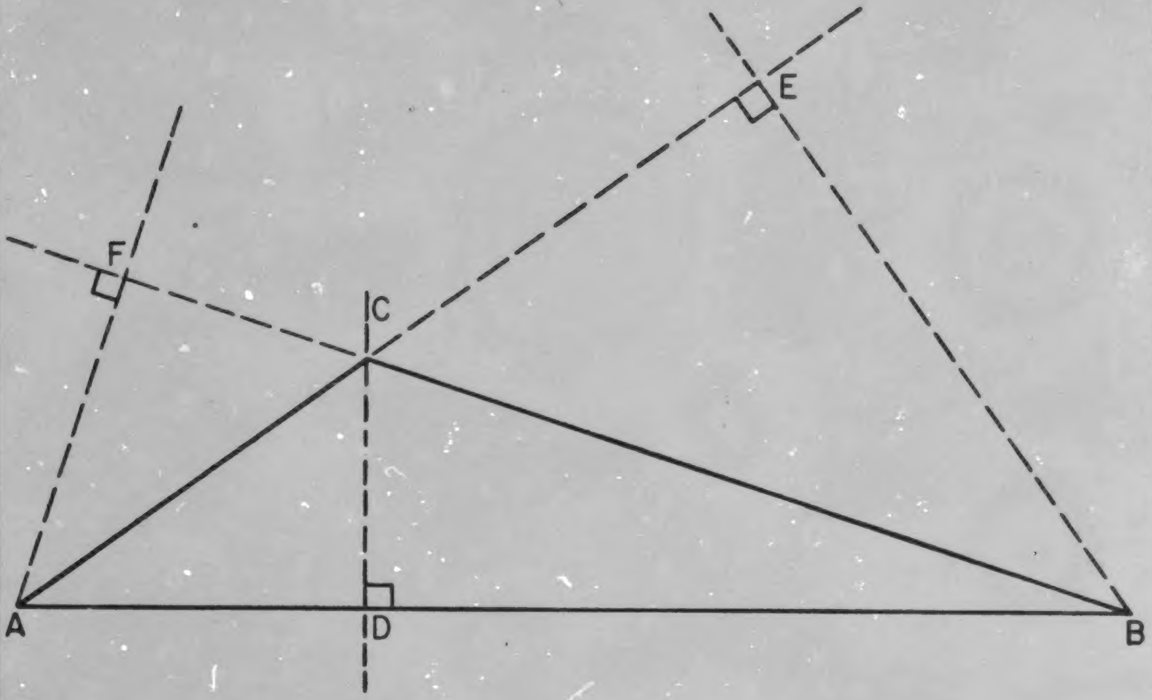
8. (a) If a figure is a quadrilateral, must it be a parallelogram? Explain your answer.

   (b) May all parallelograms be called rectangles? Explain your answer.

   (c) Can a square be called a parallelogram? Explain.

   (d) Must all parallelograms, rectangles, and squares be quadrilaterals? Explain.

9. Use the drawing below for this problem. Make all measurements to the nearest quarter of an inch.



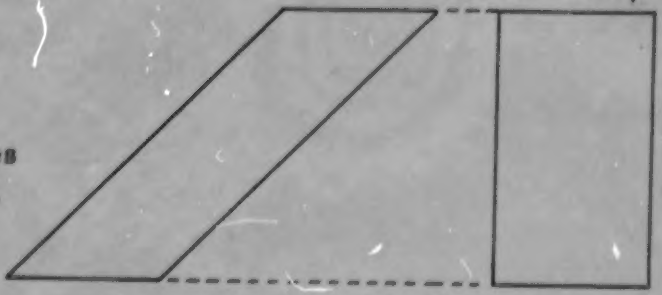
(a) Find the area of the triangle ABC by using the measures of $\overline{AB}$ and $\overline{CD}$.

(b) Find the area of the triangle ABC by using the measures of $\overline{CB}$ and $\overline{AF}$.

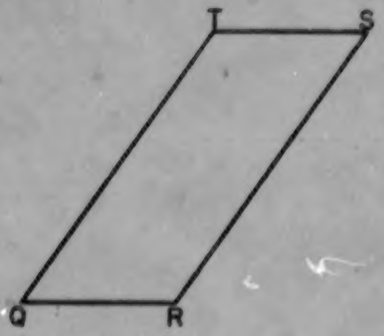
(c) Find the area of the triangle ABC by using the measures of $\overline{AC}$ and $\overline{BE}$.

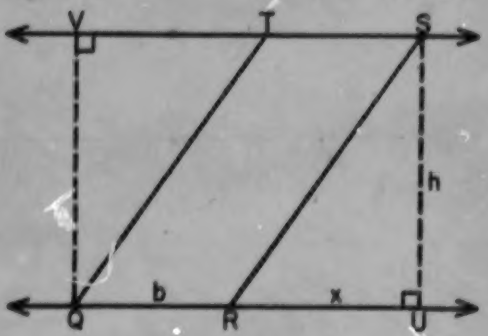
(d) Do your results in (a), (b), and (c) agree? As in problem 7, they may not be identical, but they should be close.

10. **BRAINBUSTER.** The parallelogram and rectangle shown at the right have bases with equal lengths and altitudes with equal lengths. Trace the figures on another sheet of paper. Cut out the parallelogram. Then cut the parallelogram into pieces which can be reassembled to form the rectangle.



11. **BRAINBUSTER.** Let QRST be any parallelogram not a rectangle. One possible drawing is shown below. Extend the line segments $\overline{TS}$ and $\overline{QR}$ as shown in the second figure below. At the vertices Q and S, where the angles of the parallelogram are acute, draw the perpendiculars $\overline{QV}$ and $\overline{SU}$. QUSV is a rectangle. Let b be the measure of $\overline{QR}$, h the measure of $\overline{US}$, and x the measure of $\overline{RU}$.





(a) If the measure of $\overline{QU}$ is $b + x$, what is the measure of $\overline{VS}$?

(b) What is the measure of $\overline{QV}$?

(c) What is the measure of $\overline{TS}$?

(d) What is the measure of $\overline{VT}$?

(e) What is the area of QUSV?

(f) What is the area of the triangle RUS?

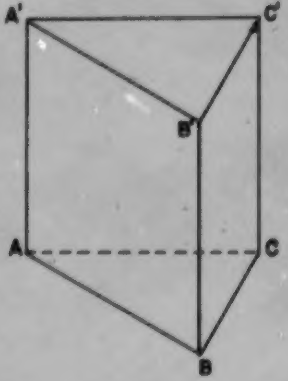
(g) What is the area of the triangle QVT?

(h) Using your answers from (e), (f), and (g) above, show that the area of QRST is given by the sentence

$$A = bh.$$

10-9. Right Prisms

In Chapter 8 you learned about rectangular prisms. Here other kinds of prisms will be introduced. Let us imagine two triangles of exactly the same size and shape lying in parallel planes. We shall say that triangles of exactly the same size and shape are congruent. Triangles ABC and A'B'C' in the figure below represent such triangles.



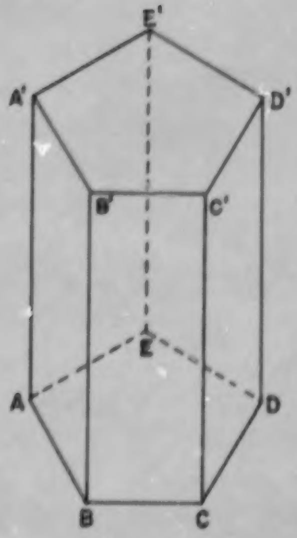
If the segments are drawn joining corresponding vertices, three quadrilaterals (four-sided polygons) are obtained. In this case they are ABB'A', BCC'B', and CAA'C'. If the triangles are so placed with respect to each other that these quadrilaterals are all rectangles, the resulting figure is called a triangular right prism.

The six points A, B, C, A', B', C' are called the vertices of the prism, the various segments shown in the figure are its edges, and the interiors of the two triangular ends and of the three rectangular sides are called its faces. To distinguish them from the interiors of the rectangular sides, the interiors of the two triangular ends are often called the bases of the prism. How many edges, vertices, and faces are there on a triangular prism?

Very likely you have seen a glass solid whose surface is a triangular prism. When held in sunlight such a solid has the effect of bending the light rays to produce the familiar rainbow effect. In fact, such solids are often called prisms.

If, in place of using triangular regions for bases, we use the regions bounded by other polygons, the resulting figures are other kinds of prisms. (Recall the definition of polygon from Section 7.) For example, look at the figure shown on the next page in which the ends are pentagons (five-sided polygons) of the same size and shape in parallel planes, and so related that the quadrilaterals ABB'A', BCC'B', etc., are all rectangles. This figure is called a pentagonal right prism.

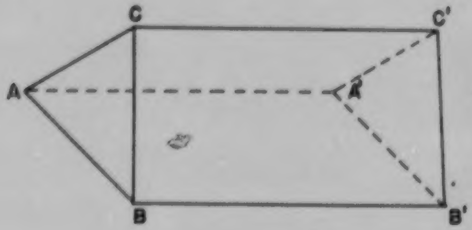
450



In general a right prism is a figure obtained from two congruent polygons so located in parallel planes that when the segments are drawn joining corresponding vertices of the polygons, the quadrilaterals obtained are all rectangles. The prism is the union of the closed rectangular regions and the two closed polygonal regions. The rectangular regions are called the faces of the prism, the segments are its edges, and the points where two or more edges meet are called vertices. The bases of the prism are the regions bounded by the original polygons.

As was indicated, the figures described here are called right prisms. Later you will meet more general prisms for which the quadrilaterals mentioned are allowed to be any parallelograms rather than necessarily rectangles. In this chapter, however, we consider only right prisms, and whenever the word prism is used it will mean right prism. The rectangular prisms discussed in Chapter 8 are, of course, simply the right prisms whose bases are the rectangular regions. These prisms have a very interesting property. A rectangular prism can be thought of as a prism in three different ways, since any pair of opposite faces can be used as bases. No other figure can be thought of as a right prism in more than one way.

In the drawings of the prisms on the preceding pages, it has been convenient to show the planes of the bases as horizontal. However, there should be no difficulty in identifying such figures when they occur in different positions. For example, the figure below represents a triangular prism with bases ABC and A'B'C', even though it is shown resting on one of its rectangular faces.



Let us consider now the problem of determining the volume of a prism. Refer to Chapter 8 and reread the discussion about the volume of a rectangular prism. This discussion showed that if the area of the base were 12 square units, then by using a total of 12 unit cubes of volume (some of which may be subdivided) we could form a layer one unit thick across the bottom of the prism. If the prism were $3\frac{1}{2}$ units high, it would take $3\frac{1}{2}$ such layers to fill the prism, or a total of $(12)(3\frac{1}{2})$ unit cubes, so the volume is 42 cubic units.

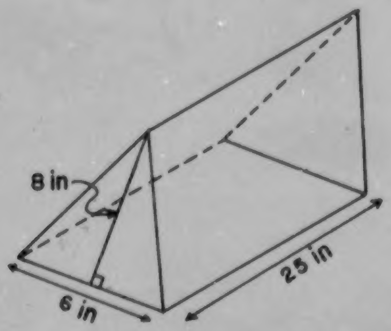
In this discussion it was not necessary to consider the actual shape of the base. In fact the same reasoning applies to finding the volume of any right prism, no matter what the shape of the base, for the volume of any right prism can be considered as consisting of a series of layers piled on each other.

We thus obtain the following conclusion:

> The number of cubic units of volume of a right prism is the product of the number of square units of area in the base and the number of linear units in the height.

In this statement, the term height means the perpendicular distance between the planes of the bases. It is the length of the segments from any vertex of one base to the corresponding vertex of the other base. Notice especially that the height is not measured vertically unless the planes of the bases happen to be horizontal. In the last figure, for example, the height is the length of any one of the segments $\overline{AA'}$, $\overline{BB'}$, or $\overline{CC'}$.

As an example let us find the volume of the triangular prism shown below.

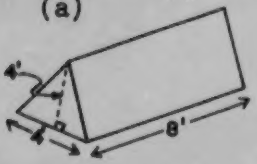


The bases are the triangular regions, and by Section 10-8 the number A of square inches in this triangular base is $A = \frac{1}{2}(6)(8) = 24$, so the area is 24 sq. in., but the number of inches in the height of this prism is 25. Thus by the statement above the number of cubic inches in the volume is $24 \cdot 25 = 600$, so the volume is 600 cu. in.

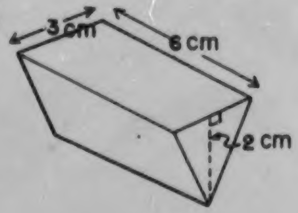
Exercises 10-9

1. Find the number of cubic units of volume for each of the prisms shown below:

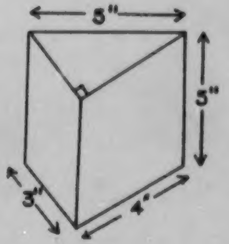
(a)



(b)

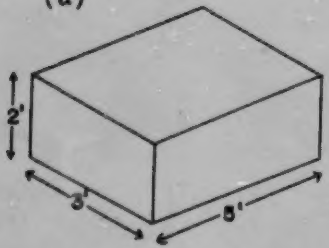


(c)

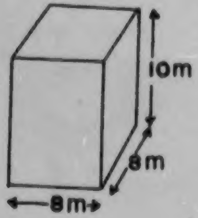


2. Find the number of cubic units of volume for each of the prisms shown below:

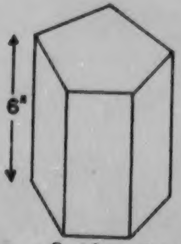
(a)



(b)



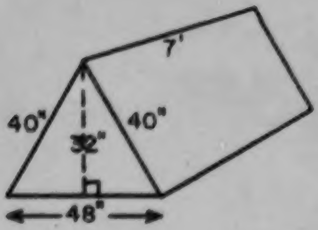
(c)



Area of the Pentagon is 21 square inches.

3. The columns in front of a building are in the shape of prisms 18 ft. high. The bases are hexagons 15 inches on a side. (A hexagon is a polygon with six sides.) If the columns are to be painted, find the number of square feet of surface for each column to be painted. (Notice that the bases--i.e., the ends--are not to be painted.)

4. A pup tent in the shape of a triangular prism is 7 ft. long. The measurements of one end are given in the drawing.



   (a) If this pup tent has canvas ends and a canvas floor how many square feet of canvas is used in its construction? (Make no allowance for seams.)

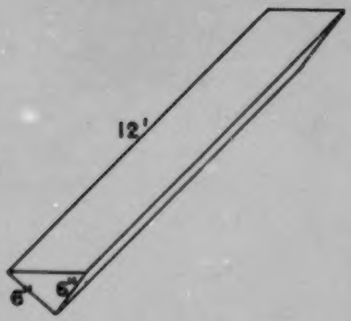
   (b) If the tent has both ends but does not have a floor, how much canvas is used?

   (c) How many cubic feet of air are in the tent?

5. If B stands for the number of square units of area in the base of a prism, and h is the number of linear units in its height, write a sentence showing how to find the number V of cubic units of volume in the prism.

6. A container in the shape of a prism is 11 inches high and holds one gallon. How many square inches are there in the base? Do you know the shape of the base? (A gallon contains 231 cu. in.)

7. A triangular prism has a base which is a right triangle with the two perpendicular sides 3 inches and 6 inches. If the prism is 20 inches high, what is the volume in cubic inches?

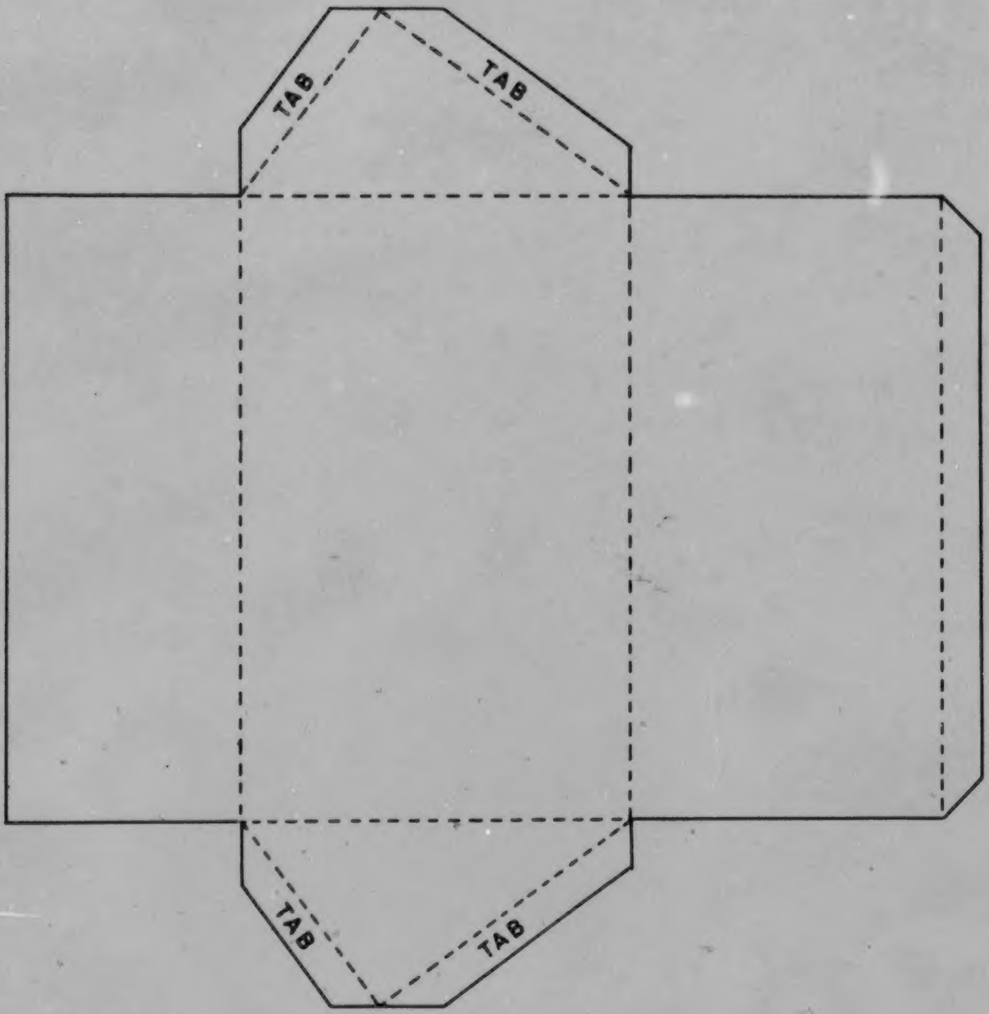


8. A trough in the shape of a triangular prism is made by fastening two boards together at right angles and putting on ends. If the inside measurements are 6 inches, as shown, and if the trough is 12 ft. long, find how many cubic feet of water it will hold.

9. Make models of triangular and pentagonal prisms from stiff paper. Either make your own patterns, or use those on the following pages.

10. How many edges, faces, and vertices are there on a triangular prism? a pentagonal prism? a hexagonal prism (6 sides)? an octagonal prism (8 sides)?

Pattern for Triangular Prism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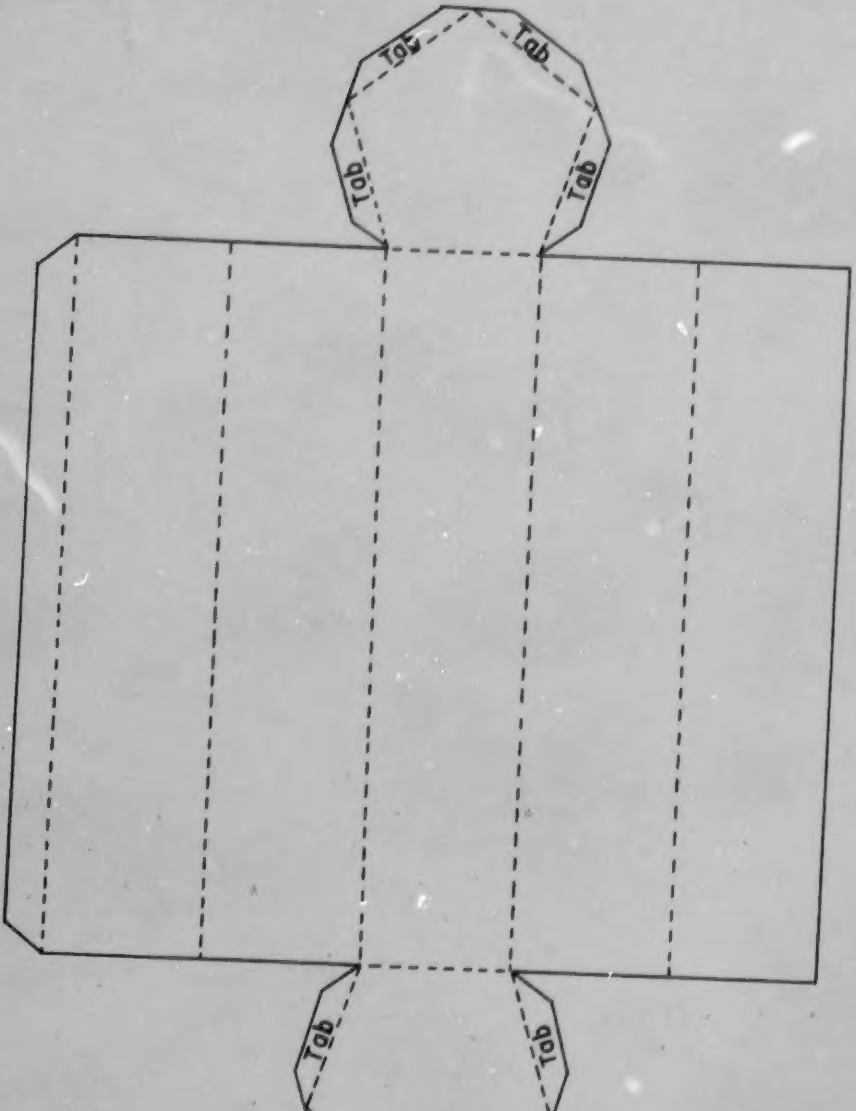
Pattern for Pentagonal Prism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10-10. Summary

Chapter 10 dealt largely with some of the relationships which exist between lines on a plane. Section 1 dealt with properties of two lines in a plane. Here you studied pairs of angles called vertical angles and observed the following property:

Property 1. When two lines intersect, the two angles in each pair of vertical angles are congruent.

In Section 1 adjacent angles and supplementary angles were also considered.

Section 2 dealt with properties of three lines in a plane, introducing the ideas of transversals and pairs of corresponding angles. Do you recall the different kinds of figures that may be formed when three lines are drawn on a plane?

In Section 3, information from Sections 1 and 2 was used to investigate two important properties.

Property 2. When, in the same plane, a transversal intersects two lines and a pair of corresponding angles are not congruent, then the lines intersect.

Property 2a. When, in the same plane, a transversal intersects two lines and a pair of corresponding angles are congruent, then the lines are parallel.

In the language of sets, Property 2 refers to two lines whose intersection is not the empty set, while Property 2a refers to two lines whose intersection is the empty set.

Converse statements were considered in Section 4. Examples were given to show that the converse of a true statement might be true, and that the converse of another true statement might be false. You were asked to write a converse for Property 2 and for Property 2a, as follows:

Converse of Property 2. If, in a plane, a transversal intersects two non-parallel lines, then corresponding angles are not congruent.

Converse of Property 2a. If, in a plane, a transversal intersects two parallel lines in a plane, then any pair of corresponding angles are congruent.

You will remember that you found these converses were both true.

In Section 5 names were introduced for different sets of triangles: isosceles, equilateral, and scalene. Scalene triangles are those having no two sides congruent. Isosceles triangles are those having at least two sides congruent, while equilateral triangles are the special set of isosceles triangles for which all three sides are congruent. In this section we discovered the following property of isosceles triangles, and found that its converse is also true.

Property 3. If two sides of a triangle are congruent, then the angles opposite these sides are congruent.

Converse of Property 3. If two angles of a triangle are congruent, then the sides opposite these angles are congruent.

In Section 6 we observed:

Property 4. The sum of the measures, in degrees, of the angles of any triangle is 180.

You obtained this property by using the inductive method of reasoning when you tore the corners from a triangular region and placed them as adjacent angles. You also learned to prove this property by the deductive method when you used previously proved properties to show that

$$m(\angle x) + m(\angle y) + m(\angle z) = 180$$

where $\angle x$, $\angle y$, and $\angle z$ represent the angles of a triangle.

The parallelogram studied in Section 7 belongs to the special set of polygons called quadrilaterals. Do you remember what quadrilateral means? A parallelogram belongs to the special set of quadrilaterals whose opposite segments lie on parallel lines.

Property 5. Opposite sides of a parallelogram are parallel and congruent.

You also learned in this section that the shortest segment from a point to a line is the one perpendicular to the line, and that two parallel lines are a constant distance apart.

In Section 8, two properties dealing with the angles of a rectangle were introduced:

Property 6a. The angles of a parallelogram at two consecutive vertices are supplementary.

Property 6b. The angles of a parallelogram at two opposite vertices are congruent.

In this section you also used certain properties of parallelograms in finding formulas for the areas of a parallelogram and a triangle:

(a) The number of square units of area in a parallelogram is the product of the number of linear units in the base and the number of linear units in the altitude to this base.

(b) The number of square units of area in a triangle is half the product of the number of linear units in the base and the number of linear units in the altitude to this base.

Section 9 dealt with volumes of right prisms, and was an extension of the work with rectangular prisms in Chapter 8. Here you learned to find the volume of the interior of any right prism:

> The number of cubic units of volume in a right prism is the product of the number of square units of area in the base and the number of linear units in the height.

10-11. Historical Note

Some of the geometric ideas in Chapter 10 were discovered by the Egyptians and Babylonians almost 4,000 years ago. For example, they knew how to find the area of a triangle and used this knowledge in surveying and measuring fields.

Thales, mentioned in Section 2, is credited with the discovery that the measures of the base angles of an isosceles triangle are equal. There is some evidence that Thales also knew that the sum of the measures in degrees of the angles in a triangle is 180.

There were many other famous Greek mathematicians. Their work made ancient Greece famous as the "Cradle of Knowledge." We will discuss only a few of these men. Pythagoras (569 ? B.C. - 500 B.C.) organized schools at Croton in southern Italy which contributed to further progress in the study of geometry. You will learn about some of the discoveries credited to him next year. Euclid (365 ? B.C. - 300 ? B.C.) became famous by writing one of the first geometry textbooks called the Elements. This textbook has been translated into many languages. It has been used in teaching geometry classes for some 2,000 years without much change. Its form has been somewhat modernized to fit present needs. All of the properties we have studied in this chapter may be found in the Elements.

From the 7th century until the 13th century very little progress was made in mathematics. From the 13th century, however, the study of geometry and other mathematics spread rapidly throughout Europe. Mathematicians began to examine new ways of studying elementary mathematics. You will learn about the work of men such as Rene Descartes (1596 - 1650, France); Blaise Pascal (1623 - 1662, France); Pierre Fermat (1601 - 1665, France); Karl Friederich Gauss (1776 - 1855, Germany); and others as you continue your studies of mathematics.

At the present time many new mathematical discoveries are being made in many parts of the world. There are still many important unsolved problems in geometry. As one example, suppose you have a lot of marbles all exactly the same size. How should you pack them in a large container in the best possible way, that is, so as to get as many marbles as you can into the given volume? Nobody knows for sure. We know a very good way to pack the marbles, but no one has ever proved that it is the best possible way.

Originally geometry meant a study of earth measure. The word "geometry" comes from two Greek words, "geo" meaning earth and "metry" meaning measure. Through the years geometry has been thought of as a study of such elements as points, lines, planes, space, surfaces, and solids. These elements are used to describe the shape, size, position, and relations among objects in space.

Chapter 11

CIRCLES

One of the most common simple closed curves is the circle. No matter where you are, it is probable that you will be able to see one or more objects which suggest a circle. Can you find some in your classroom now? Find as many as you can in your home. There may be a traffic "circle" near your school. Do you think a circle is a good figure to use in this way?

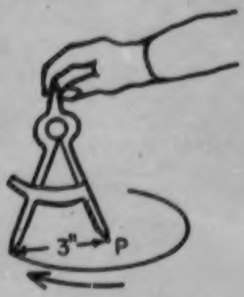
In Chapter 4, you learned about simple closed curves. In later chapters you studied some of the characteristics of several kinds of simple closed curves, such as parallelograms, rectangles, and triangles. In this chapter, you will study some of the properties of a circle as a mathematical figure.

You already know that a line or a simple closed curve may be thought of as a set of points. Let us see how we may describe a circle as a set of points.

11-1. Circles and the Compass

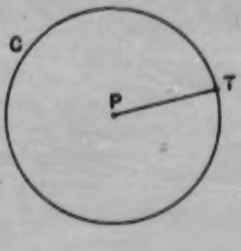
Choose a point near the center of a sheet of paper. Label the point P. Then, using your ruler and a pencil, mark at least ten points, each at a distance of 3 inches, from P. What figure do these points suggest? If you have located the ten points in widely different directions from P, they should suggest a circle.

To draw a complete circle, use a compass. Some of you are already familiar with the compass as a device for drawing circles. (An instrument for indicating which direction is north is also called a compass, but we are not concerned with that kind of compass here.) To draw a circle with a compass, adjust the arms



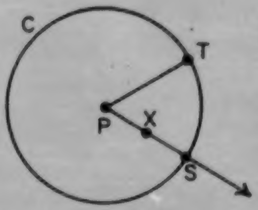
of the compass so that the distance between the sharp point and the pencil tip is the desired distance, which in this case is 3 inches. Place the sharp point of the compass on the point P, which you located previously on your sheet of paper. Do not move the sharp point of the compass while making the drawing! Pivot the compass about the point P, making the pencil tip draw a curve. Hold the compass between the thumb and forefinger as shown in the drawing. When doing this, lean the compass in the direction which you are pivoting the pencil tip. When the pencil tip returns to the starting point, you have completed your drawing of a circle. Is a circle a simple closed curve? Does your drawing fit the description of a simple closed curve given in Chapter 4? If you located the ten points accurately, your drawing of the circle should pass through each of the points.

In your circle, point P is the center of the circle. Choose any one of the points you located at a distance of 3 inches from P. Label this point T. Draw a segment joining P and T as shown in the figure at the right. $\overline{PT}$ has a measurement of 3 inches. This circle may be called "circle P," meaning a circle which has as its center the point P. Sometimes we name a simple closed curve by a letter, as "circle C." When we say "circle C," C is not just one point of the circle. It represents the entire circle.



The segment $\overline{PT}$ is called a radius of the circle. A radius is any line segment which joins the center P to a point on the circle. Draw a second radius. Call it $\overline{PY}$. When we speak of more than one radius, we say "radii." $\overline{PT}$ and $\overline{PY}$ are radii of the circle. How many such radii of the circle may there be?

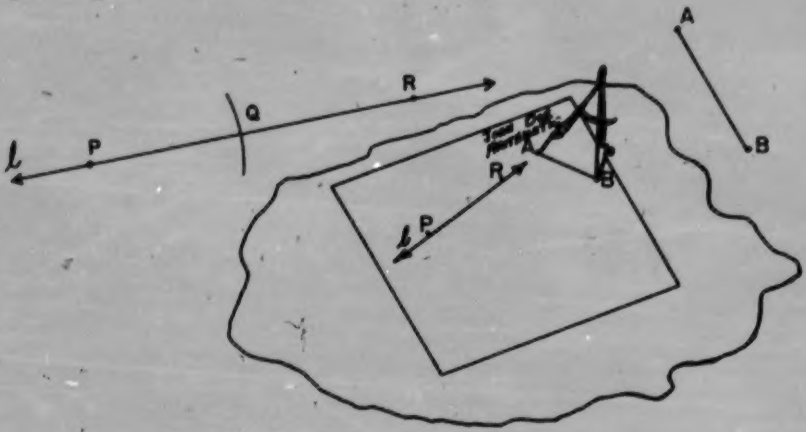
We use the word "radius" in another way too. It means the distance from the center to any point on the circle. The radius of the circle you drew is 3 inches. There is just one distance which is "the radius of a circle, but "a" radius may be any line segment having the center of the circle and a point on the circle as its endpoints.



Look at your drawing again. Choose any point inside the circle and call it point X. Draw the ray $\overrightarrow{PX}$. On the ray $\overrightarrow{PX}$, measure a distance 3 inches from P. The point you obtain should be on the circle. Call this point S. Is $\overline{PS}$ a radius of the circle?

We can now describe a circle as a set of points. The circle, with center P and radius r units, is the set of all points in a plane at a distance r from P.

The compass has a second use. It is used for transferring distances. Draw a line ℓ and label two points, P and R, on ℓ. Draw a segment AB anywhere on your paper. Without using a ruler to measure the length of $\overline{AB}$, we wish to find a point Q on $\overline{PR}$ such that the length of $\overline{PQ}$ is the length of $\overline{AB}$. Adjust the arms of your compass as shown in the drawing below, so that when you put the sharp point at A, the pencil tip is at B. Then, without changing the opening of the arms, lift the compass and place the sharp point at P. Draw a small part of a circle crossing the ray $\overrightarrow{PR}$, still without changing the opening of the arms of the compass.



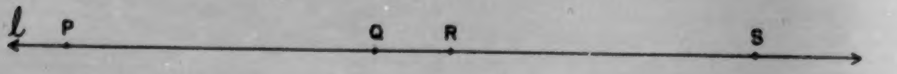
Call the point of intersection Q. Then the length of $\overline{PQ}$ is the length of $\overline{AB}$.

Exercises 11-1

In the following problems, you will have practice in using your compass to draw circles and to transfer distances. Read the directions carefully. Label each point, circle, or line segment in your drawing before going on to the next direction.

1. (a) Label a point P on your paper. Draw a circle with center P and radius 7 cm. Call it circle C.
   (b) Label Q a point on circle C. Draw a circle with center at Q and a radius 3.5 cm.
   (c) Draw a circle with center at P and radius 3.5 cm.
   (d) What does the intersection of the last two circles seem to be?

2. In this problem, you are to refer to the drawing below. Follow the directions listed below the drawing.



   (a) Near the center of a sheet of paper, draw a vertical line about 6 inches in length. Label the line m. Label a point A near the lower part of the line.
   (b) Use your compass to locate a point B above A on line m, so that AB = PQ. (Note that "AB", with no symbol above it, means "the measure of segment $\overline{AB}$.")
   (c) Locate point E on line m, so that E is above A and AE = PR.
   (d) Locate point F above A on line m so that BF = RS.
   (e) With B as center and $\overline{BF}$ as radius, draw a cricle.
   (f) If your drawing is accurate, there will be two labeled points on the circle. Name the points.

3. (a) Draw two intersecting lines which are not perpendicular, and call the lines ℓ_1 and ℓ_2. (Notice that we have named both lines with the same letter, but have written numerals after the letters, and a little lower. You will recall that such numerals are called "subscripts." ℓ_1 is read "ℓ sub-one," or sometimes just "ℓ one." "ℓ_2 is read "ℓ sub-two.")

   (b) Call the point of intersection of ℓ_1 and ℓ_2 point B. With B as center and radius 1 inch draw a circle.

   (c) Label the intersections of line ℓ_1 with the circle R and S, and the intersections of line ℓ_2 with the circle T and Y.

   (d) Draw $\overline{RT}$, $\overline{RY}$, $\overline{ST}$, and $\overline{SY}$. What kind of figure does RTSY seem to be?

4. Draw a line ℓ and label points X and Y about 1 inch apart on ℓ.

   (a) Draw the circle C_1 which has its center at X and passes through Y.

   (b) Draw the circle C_2 which has its center at Y and passes through X.

   (c) Label Z as the other intersection of circle C_2 and line ℓ.

   (d) Draw the circle C_3 which has its center at Z and passes through X.

   (e) What is $C_1 \cap C_3$?

   (f) What is $C_2 \cap C_3$?

*5. (a) Draw two intersecting lines ℓ_1 and ℓ_2. Label as B the intersection of ℓ_1 and ℓ_2.

(b) Label A a point of ℓ_1 and C a point of ℓ_2, so that the length of $\overline{BA}$ is not the length of $\overline{BC}$.

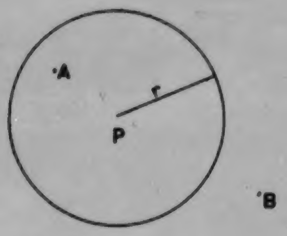
(c) Use your compass to mark a point A_1 on $\overrightarrow{AB}$ such that A_1 is not on $\overrightarrow{BA}$ and $A_1B = AB$.

(d) Mark a point C_1 on $\overrightarrow{CB}$, but not on $\overrightarrow{BC}$, such that $BC_1 = BC$.

(e) Draw $\overline{AC}$, $\overline{AC_1}$, $\overline{A_1C}$, and $\overline{A_1C_1}$. What kind of figure does ACA_1C_1 appear to be?

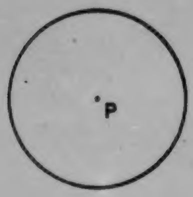
11-2. Interiors and Intersections

A circle is a simple closed curve. Consequently it has an interior and an exterior. Suppose we have a circle with center at P and with radius r units. A point such as A inside the circle is less than r units from the center P, while a point outside the circle, such as B, is more than r units from P. Thus it is easy, in the case of the circle, to describe precisely what its interior and its exterior are. The interior is the set of all points at distance less than r units from P. The exterior is the set of all points at distance greater than r units from P.

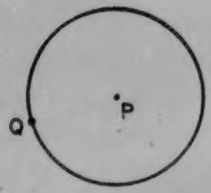


We have frequently worked with the notion of the intersection of two sets. A circle is an example of a set of points. Consequently we may raise questions about intersections involving circles.

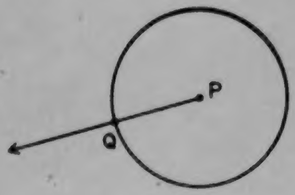
Let us choose a point P and a line segment of any length. Let us draw the circle with center at P and radius equal to the selected segment. With your compass, make the drawing on a piece of paper. Your picture should look like the figure at the right.



Next, let us choose any point Q on the circle. (After we make our choice, the drawing will look like the figure at the right.)

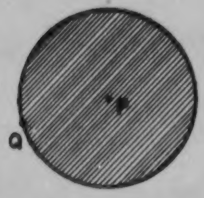


How many points on the circle are also on the ray $\overrightarrow{PQ}$? To answer this question, you may find it easier to draw the ray on your paper. It should look like the drawing at the right. (In general you should acquire the habit of putting on paper whatever you need in order to understand the ideas you are studying. Notice how we have done this, step by step, in our discussion. Follow this suggestion throughout the remainder of the chapter.)



With the same situation as in the previous paragraph, how many points on the circle are also on the ray $\overrightarrow{QP}$? (Did you draw the ray $\overrightarrow{QP}$ before you attempted to answer?) Do you feel the need to label one or more points which we have not yet named? Can you describe carefully (in words) the location of the extra point which you believe ought to be named?

Now shade lightly the interior of the circle as shown. What is the union of the circle and its interior? In Chapter 8, the union of a simple closed curve and its interior was called a "closed region." The union of the circle and its interior is a circular closed region. The set of all points which are either on or inside the curve is the union of the circle and its interior. Another way of thinking about the union of the circle and its interior is the following:

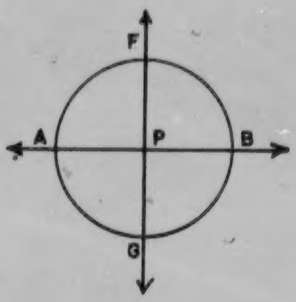


> The union is the set of all points whose distance from the center P is either the same as or less than the radius of the circle.

What is the intersection of the circle and its interior? No point of the circle also lies in the interior of the circle. We say that the intersection of the circle and its interior is the empty set.

Let Y represent the union of the circle and its interior. What is the intersection of the set Y and the line $\overleftrightarrow{PQ}$. In what way is $Y \cap \overleftrightarrow{PQ}$ quite different from the intersection of $\overleftrightarrow{PQ}$ and the circle?

In the figure at the right, point P is the center of the circle. The four points A, F, B, and G lie on the circle. The intersection of $\overleftrightarrow{AB}$ and $\overleftrightarrow{FG}$ consists of the point P, and $\overleftrightarrow{AB}$ is perpendicular to $\overleftrightarrow{FG}$. Follow these directions



in making a copy of the diagram on a sheet of paper:

(1) With compass, draw the circle. Mark the center and label it P.

(2) Draw a line through P. Label the points of intersection with the circle A and B.

(3) Use a protractor to obtain a right angle, and draw a line perpendicular to $\overleftrightarrow{AB}$. Label the points of intersection of the circle and the line F and G.

(4) Shade the half-plane which contains F and whose boundary is $\overleftrightarrow{AB}$. (A colored pencil or crayon is useful for shading. The shading should be done lightly. If you do not have a colored pencil or crayon, shade lightly with your pencil.)

Let us call this particular half-plane H. Now answer the following:

(a) What is the intersection of the half-plane, H, and the circle?

(b) Does point A belong in this intersection? Does G ? Does F ? Does P?

(c) Can you count all the points that belong to this intersection? Explain why, or why not.

Choose two points on the circle which also belong to the interior of the angle BPF. Label these points M and N. Similarly, choose one point, which we will call K, which lies on the circle and also in the interior of angle APF.

(d) What is the intersection of the circle and angle MKN?

(e) What is the intersection of the interior of the circle and the interior of the angle MKN?

Exercises 11-2

1. Let C be a circle with center at P and radius r units. Let S be any other point in the same plane. (Are you drawing the figure, step by step, as the problem is describing it?)

   (a) How many points belong to the intersection of the circle C and the ray $\overrightarrow{PS}$?

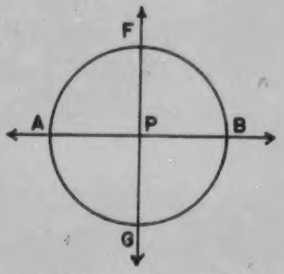
   (b) How many points belong to the set $C \cap \overleftrightarrow{PS}$?

   (c) Do your answers to (a) and (b) depend upon where you chose the point S?

   (d) How many points belong to the set $C \cap \overline{PS}$? Does this answer depend on the choice of S? If so, how?

2. In a plane, can there be two circles whose intersection consists of just one point?

3. Choose two points and label them P and Q. Draw two circles with center at P such that Q is in the exterior of one circle and in the interior of the other. Label the first circle C and the second circle D.



4. In the figure at the right, let the half-plane on the F-side of $\overleftrightarrow{AB}$ be called H. Let J be the half-plane on the B-side of $\overleftrightarrow{FG}$.

   (a) What is the set $H \cap J$?

   (b) What is the intersection of all three of the sets, J and H and the circle?

   (c) Does the diagram suggest to you that the circle has been separated into what might be called quarters? If so, can you describe several of these portions?

   (d) Can you find a portion which might be called half of the circle? Can you describe it in the language of intersections or unions of sets? Can you identify several such portions? More than two?

5. Choose two distinct points P and Q. Draw the circle with center at P and with the segment $\overline{PQ}$ as a radius. Then draw the circle with center at Q and with P on the circle.

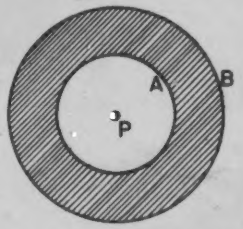
   (a) What is the intersection of these two circles?

   (b) Can you draw a line which passes through every point of the intersection of the two circles? Can you draw more than one such line? Why?

   (c) In your picture shade the intersection of the interiors of the two circles. (If you have a colored pencil handy, use it for shading; if you do not, use your ordinary pencil and shade lightly.)

   (d) (In this part, use a different type of shading or, if you have one handy, use a pencil of a different color.) Shade the intersection of the interior of the circle whose center is P and the exterior of the circle whose center is Q. (Before doing the shading of the intersection, you may find it helpful to mark separately the two sets whose intersection is desired.)

   (e) Make another copy of the picture showing the two circles, and on it shade the union of the interiors of the two circles.



6. The two circles shown at the right lie in the same plane and have the same center, P. Circles having the same center are called concentric circles. (On an automobile wheel, the rim of the wheel and the circular edge of the tire are examples of concentric circles. If you throw a stone into a pool of water, the circular waves formed are also examples of concentric circles.)

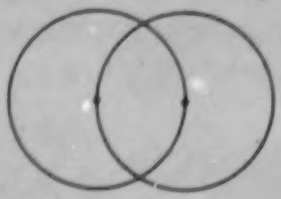
   (a) Describe the intersection of circle A and circle B.

   (b) Give a word description of the shaded region, using such words as "intersection," "exterior," and so on.

[sec. 11-2]

*7. Refer to the figure in Problem 6. How may the intersection of the exteriors of the two circles be more simply described?

*8. In the drawing at the right, the center of each circle lies on the other circle. Copy the figure on your paper. Shade the union of the exteriors of the two circles.

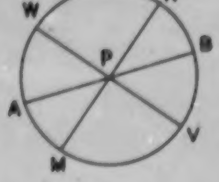


*9. Choose two distinct points P and Q. Draw the line $\overleftrightarrow{PQ}$. Draw the circle with center at P and with Q on the circle. Draw the circle with center at Q and with $\overline{QP}$ as a radius. Draw a line which passes through each point of intersection of the two circles, and call the line ℓ. By inspecting the diagram, make an observation about a relationship between the line ℓ and the line $\overleftrightarrow{PQ}$. Use your protractor to check your observation.

11-3. Diameters and Tangents

We have seen that the word "radius" can be used in two different ways. By way of review, a radius of a circle is one of the segments joining a point of the circle and the center. The length of one of these segments is the radius of the circle.

The word diameter is closely associated with the word radius. A diameter of a circle is a line segment which contains the center of the circle and whose endpoints lie on the circle. For the circle represented by the figure at the right, three diameters are shown; $\overline{AB}$, $\overline{MN}$, and $\overline{VW}$. (A diameter of a circle is the longest line segment that can be drawn in the interior of a circle such that its end-points are on the circle.) How many radii are shown in the figure?



A set of points which is a diameter may be described in another way. A diameter of a circle is the union of two different radii which are segments of the same line. How does the length of a diameter compare with the length of a radius?

The length of any diameter of a circle is also spoken of as the diameter of the circle. The diameter is a distance, and the radius is a distance.

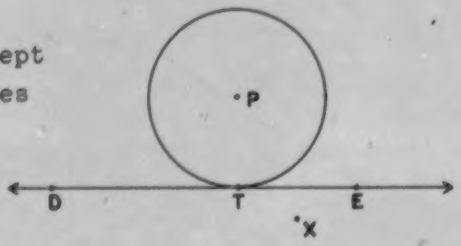
The measure of the diameter is how many times the measure of the radius? If we choose any unit of length, and if we let r and d be the measures of the radius and the diameter of a circle respectively, then we have this important relationship:

$$d = 2\ r.$$

What replacement for the question mark makes the following number sentence a true statement?

$$r = ?\ d.$$

The line and the circle in the figure on the right remind us of a train wheel resting on a track, except that the flange (or lip) which guides the train is not shown. How many points are on the circle and also on the line? There is only one such point, the one labeled T. We say that the line is tangent to the circle. The single point of their intersection is the point of tangency. In this drawing, T is the point of tangency. Another way of describing a point of tangency is to say that it is the only point of the circle which is also on the line. Now answer the following questions:



(1) In the figure, the line $\overleftrightarrow{DE}$ separates the plane DPX into two half-planes. How can you describe the intersection of the half-plane containing X and the circle?

(2) What is the intersection of the half-plane containing P and the interior of the circle?

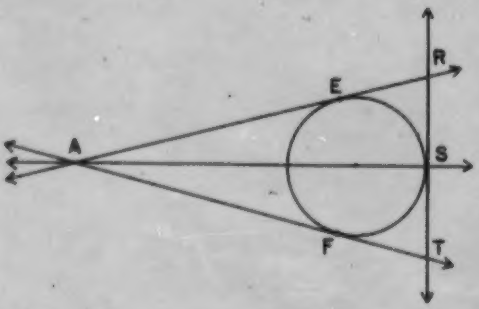
(3) Can you draw a circle and a line in the same plane such that their intersection is the empty set? Explain.

(4) Can you draw a circle and a line such that their intersection contains exactly four points? Explain.

5. (a) In the drawing at the right, how many lines are shown?

   (b) How many of these lines are not tangent to the circle?

   (c) Name each point of tangency.



6. Suppose that we are given a circle and a point on the circle.

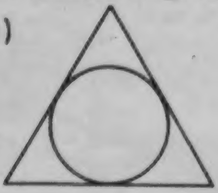
   (a) How many radii of the circle contain the given point?

   (b) How many lines, tangent to the circle, contain the given point?

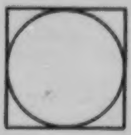
Exercises 11-3.

1. How many tangents do you find in each of the following?

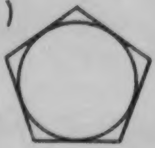
(a)



(b)



(c)



2. Look for examples which represent the idea of a circle and a line tangent to the circle, that is, of a line and a circle whose intersection consists of a single point. Describe these examples.

3. The diameters of certain circles are listed. For each circle, find the distance from the center of the circle to any point on the circle.

(a)	42 cm.	(d)	4 yd.
(b)	28 in.	(e)	30 ft.
(c)	10 ft.		

4. Find the diameters for each of the circles where the distance from the center of the circle to a point on the circle is as follows:

(a)	6 in.	(d)	5 ft.
(b)	3 m.	(e)	$3\frac{1}{2}$ ft.
(c)	17 cm.		

5. Draw a circle C with center at the point P. Draw three diameters of C. Draw a circle with center at P whose radius is equal to the diameter of C.

6. (Warning: this problem requires very careful handling of the compass.)

   (a) Near the middle of a sheet of paper, mark a point Q.

   (b) Draw a circle C with center at Q, and radius approximately 2 inches. Keep the opening of the compass arms fixed at the radius selected until the drawing is completed.

   (c) Mark a point U on the circle C.

   (d) With the compass, find a point V on the circle C such that $\overline{UV}$ has the same length as the radius $\overline{QU}$.

   (e) Again, with the compass, find a third point W on C such that $\overline{VW}$ has the same length as $\overline{QU}$.

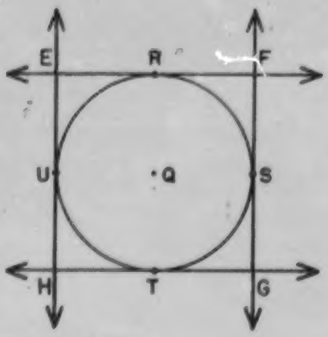
(f) Continue on around, locating points X, Y, Z on C (use compass three times) such that the length of each of the segments $\overline{WX}$, $\overline{XY}$, $\overline{YZ}$ is the same as the radius of the circle.

(g) Now compare the length of the segment $\overline{ZU}$ with the radius of the circle.

If you had a fine quality compass and if you were able to perform the construction with great care, the simple closed curve UVWXYZ would represent a hexagon. (In such a figure, if the sides are congruent and the angles are congruent, the figure is called a regular hexagon. Your drawing should look like a regular hexagon. Note that the segments which are sides of your hexagon are a little shorter than one-sixth of the circle, that is, the arc which has endpoints the same as a segment.)

(h) What name might you give to each of the segments $\overline{UX}$, $\overline{VY}$, and $\overline{WZ}$?

In the diagram the point Q is the center of the circle and the center of the square EFGH.



(a) What is the intersection of the circle and the square?

(b) What is the intersection of the line $\overleftrightarrow{GH}$ and the circle?

(c) What new name have we given to the point T?

(d) How many lines are tangent to the circle?

(e) Name all the points of tangency.

8. On your paper make a sketch of the diagram for Problem 7. (A careful drawing is not needed here.) On your copy, draw the quadrilateral RSTU.

   (a) How many sides of this quadrilateral are segments of lines tangent to the circle?

   (b) What is the intersection of the interior of the circle and the exterior of the square EFGH?

   (c) Describe carefully the intersection of the exterior of the circle and the interior of the square EFGH.

   (d) What is the intersection of the interior of the circle and the exterior of the quadrilateral RSTU? Explain your answer by shading the correct region on the figure.

*9. Refer again to the carefully drawn figure in Problem 7.

   (a) Do you think that all three of the points R, Q, T lie on one line?

   (b) Do the three points U, Q, S appear to lie on one line?

   (c) Make an estimate of the size of the angle QTG. Check your estimate with a protractor.

   (d) Make an observation about a relationship between the line $\overleftrightarrow{QS}$ and the line $\overleftrightarrow{FG}$. Check your observation with a protractor.

*10. Draw any circle and any line tangent to the circle. Draw also the line which joins the center of the circle and the point of tangency.

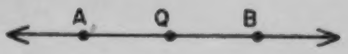
   (a) Do you believe there is an important relationship between these two lines? If so, what is it?

   (b) How many radii of a circle may be drawn having one point of tangency as an endpoint?

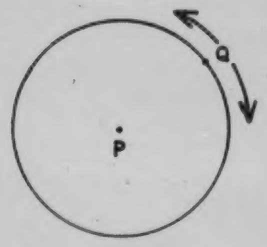
*11. We know from the description of a circle that all radii are congruent. How can we use this fact to show that by our definition of a diameter all diameters of a particular circle are congruent?

11-4. Arcs

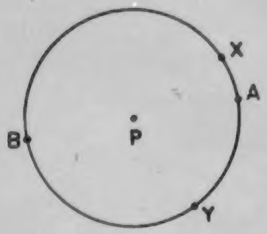
In Chapter 4 we learned that a single point on a line separates the line into two half-lines. Using this idea of separation, we say that on a line, a single point determines two half-lines. Recall that if A, B, and Q are points on a line, as shown in the figure at the right, Q is considered to be "between" A and B. On a line, the idea of "betweenness" and the idea of "separation" are closely tied together.

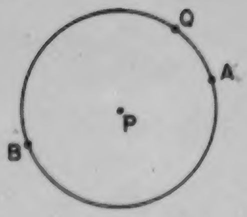


Does a single point on a circle separate the whole circle into two parts? Does point Q separate the circle at the right into two parts? If we start at Q and move in a clockwise direction we will, in due time, return to Q. The same is true if we move in a counter-clockwise direction. A single point does not separate a circle into two parts.



In the figure at the right, the two points, X and Y, separate the circle into two parts. One of the parts contains the point A. The other part contains B. No path from X to Y along the circle can avoid at least one of the points, A and B. Thus, we see it takes two different points to separate a circle into two distinct parts.



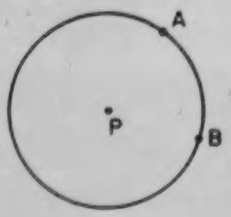


Refer to the circle at the right showing points A, B, and Q on the circle. Is point A between B and Q? Is point Q between A and B? Is point B between A and Q? Since we can move in either a clockwise or counter-clockwise direction on the circle, the answers to these three questions are yes. Unlike a line where betweenness and separation are closely related, on a circle, or other simple closed curve we can observe these two ideas:

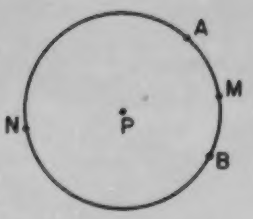
(1) A single point does not separate the curve into two parts.

(2) Separation and betweenness are not closely related notions for simple closed curves.

In earlier discussions of geometry we referred to parts of lines as line segments. It will be necessary for us to consider parts of circles. A part of a circle is called an arc. In the drawing at the right, points A and B separate the circle into two parts. Each of these parts, together with the points A and B, is an arc. A and B are the endpoints of the arc. Any two distinct points on a circle determine two different arcs having these two points as endpoints. In the drawing above, the arc, starting at point A and moving clockwise to point B is shorter than the arc starting at point A and moving counter-clockwise to point B.

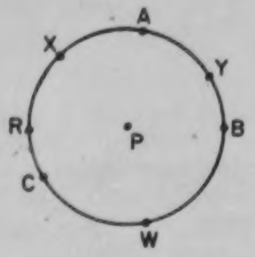


With only two points on a circle, such as A and B, we cannot easily identify one of the two arcs determined by A and B. In the drawing at the right, we have marked and labeled a point between each of the two end-points on each of the arcs. These two points are conveniently located somewhere near the middle of each of the arcs. We use the symbol "$\frown$" to represent the word "arc". Thus, $\overset{\frown}{AMB}$ represents the arc containing point M. $\overset{\frown}{ANB}$ represents the arc which contains N. In place of $\overset{\frown}{AMB}$ we can use $\overset{\frown}{BMA}$. What other symbol represents the same arc as $\overset{\frown}{ANB}$? What point is contained in the arc $\overset{\frown}{MAN}$?



Exercises 11-4a.

1. Using the drawing on the right, identify the shortest arc containing the following points, where the points are not endpoints of the arc.



(a) A
(b) B
(c) C
(d) R
(e) X
(f) Y

2. Using the drawing for Problem 1, name the point or points which are not endpoints, of each of the following arcs.

(a) $\overset{\frown}{BWC}$
(b) $\overset{\frown}{ABC}$
(c) $\overset{\frown}{WAX}$
(d) $\overset{\frown}{CAB}$
(e) $\overset{\frown}{XBY}$
*(f) $\overset{\frown}{AC}$

3. What are the endpoints for these arcs from the drawing in Problem (1)?

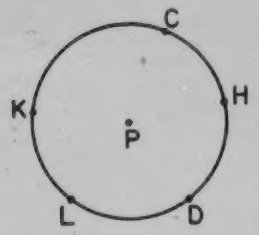
   (a) $\overset{\frown}{AYB}$ (d) $\overset{\frown}{YWC}$

   (b) $\overset{\frown}{AXR}$ (e) $\overset{\frown}{WRA}$

   (c) $\overset{\frown}{ACB}$ (f) $\overset{\frown}{BAC}$

4. In answering the previous question, was it necessary to see the drawing? Explain.

5. Use the drawing at the right in answering the following:



   (a) On the circle, is there one specific point between C and D? Explain.

   (b) Is there one specific point between C and D on $\overset{\frown}{CHD}$? If so, name the point.

   (c) Point L separates $\overset{\frown}{CLH}$ into two arcs. Name these two arcs.

   (d) Does point L separate the circle into two arcs? Explain why, or why not.

6. Use your answers in Problem 5, above, to answer the following:

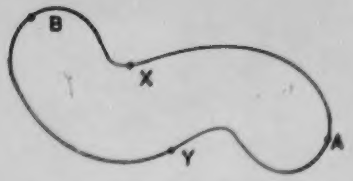
   (a) Does a point on an arc separate the arc into two arcs?

   (b) On an arc, must a point, which is not an endpoint of the arc, be between two points of the arc?

   (c) Does an arc have a "starting" point and a "stopping" point? If so, what are they called.

   (d) For an arc are the notions of betweenness and separation more like those of a line segment or of a circle?

7. In the figure at the right the pair of points A and B separate the points X and Y. Which points, if any, do the pair of points listed below separate?



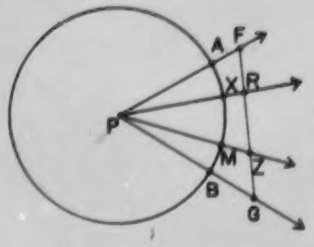
(a) B, Y

(b) A, Y

(c) X, Y

(d) A, X

*8. In the figure shown at the right, $\overset{\frown}{AMB}$ is associated with the interior of $\angle APB$. $\overline{FG}$ connects a point of $\overrightarrow{PA}$ with a point of $\overrightarrow{PB}$.



(a) Are points A and F both on $\overrightarrow{PA}$?

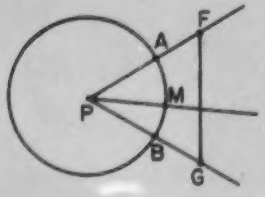
(b) Does point F on $\overline{FG}$ correspond to point A on $\overset{\frown}{AMB}$?

(c) For each ray from P and passing through $\overset{\frown}{AMB}$ is there a point on $\overline{FG}$ which is also on the ray?

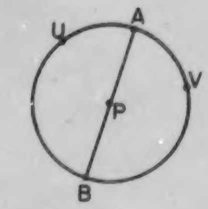
(d) Is there a one-to-one correspondence between the points on $\overset{\frown}{AMB}$ and the points on $\overline{FG}$? If so, describe the correspondence.

11-5. Central Angles

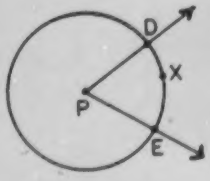
Arcs have some properties similar to the properties of line segment. In the figure shown at the right, there is a natural one-to-one correspondence between the set of points of $\overset{\frown}{AMB}$ and the set of points of $\overline{FG}$. A point may separate a segment into two parts. Similarly, an arc may be separated into two arcs by a point which is on the arc, but is not an endpoint of the arc. As with a segment, an arc has a "starting" point and a "stopping" point, the two endpoints. Although arcs are parts of circles, an arc has some properties which are not the same as the properties of a circle.



In the figure shown at the right, $\overline{AB}$ is a diameter of circle P. Two arcs are determined by the endpoints A and B. Such arcs are very special arcs and are given a special name. They are called semicircles. A semicircle is an arc determined by the endpoints of a diameter of the circle. $\overset{\frown}{AVB}$ is a semicircle in the figure shown. Can you name another semicircle in this figure? Is $\overset{\frown}{BUA}$ a semicircle?



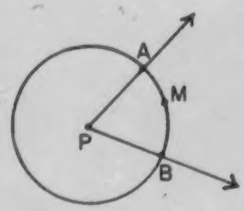
The endpoints of a semicircle and the center of the circle are on a straight line. This, however, is not true for all arcs. In the drawing at the right, the endpoints of $\overset{\frown}{DXE}$ are not on a straight line passing through the center of the circle. D and E are on the rays $\overrightarrow{PD}$ and $\overrightarrow{PE}$. Angle EPD has its vertex at the center of the circle. We call such angles central angles. A central angle is an angle



having its vertex at the center of a circle. Such angles are measured in the same way as other angles. The unit of angle measure is an angle of one degree. In the figure, the measure of angle EPD is about 85.

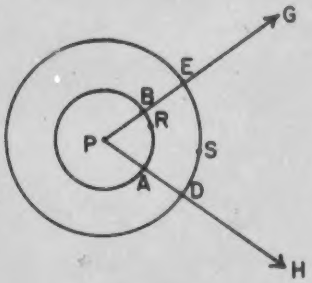
In working with arcs, we find it necessary to compare one arc with another. Therefore, we will find it convenient to devise a method for measuring arcs. We think of a circle divided into 360 congruent arcs, that is, each of the 360 such arcs in a particular circle has the same measure. Each such arc determines a unit of arc measure. We call this unit one degree of arc.

From the center of the circle, rays which pass through the endpoints of such an arc determine a central angle. We think of a degree of arc as being determined by a central angle which is a unit angle of one degree. In the figure at the right, if the measure of $\overset{\frown}{AMB}$ in degrees of arc is 80, then the measure of $\angle APB$ in angle degrees is 80. The symbol for a degree of arc, "$^\circ$", is the same as that for the angle degree. For measuring degrees of arc, we can use an instrument with a circular shape containing 360 congruent arcs. Most such instruments, called protractors, show just half of this scale.



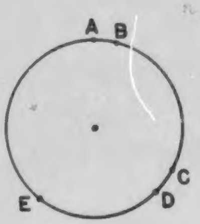
On the set of numbered rays of the angle scale, the numbers 0 and 180 correspond to opposite rays, that is, rays which lie on the same line and have the same endpoint. The union of these two rays forms a line whose intersection with the circle is the pair of **endpoints of a diameter**. These two points determine a special arc, a **semicircle**, which we have mentioned earlier in the chapter. Thinking of this angle scale we can think of the **semicircle** having an arc degree measurement of 180° since it would consist of 180 one degree arcs. Corresponding to the special kind of arc which we call a semicircle is a special kind of central angle with a measurement of 180°. Some people find it convenient to speak of the central angle of 180° as a "straight angle." (Why does "straight angle" not agree with our definition of angle?)

In the figure at the right are two circles having a common center, P. The circles are in the same plane. Such circles we have called concentric circles. The two arcs, $\overset{\frown}{ARB}$ and $\overset{\frown}{ESD}$ have the same central angle, $\angle$ GPH. Therefore, $\overset{\frown}{ARB}$ and $\overset{\frown}{ESD}$ must have the same arc measure. If the angle measure of $\angle$ BPA is 70, then the arc measure of $\overset{\frown}{ARB}$ is 70. The arc measure of $\overset{\frown}{ESD}$ must also be 70. However, $\overset{\frown}{ARB}$ appears shorter than $\overset{\frown}{ESD}$. Remember that arc measure is not a measure of length. Two arcs may have the same arc degree measure but have different lengths. The reason will be more apparent after you have studied the remainder of this chapter.



Exercises 11-5

1. In the figure at the right determine with the use of your protractor the measure of the following arcs. Indicate your results with correct use of symbols, for example $m(\overset{\frown}{AB}) = 15$.



(a) $\overset{\frown}{ABC}$

(b) $\overset{\frown}{ABCD}$

(c) $\overset{\frown}{DE}$

(d) $\overset{\frown}{BCD}$

(e) $\overset{\frown}{CDE}$

2. Construct a circle with radius approximately $1\frac{1}{2}$ inches. In this exercise mark off the points in a counter-clockwise path around the circle after starting anywhere on the circle with A. Mark off and label arcs with the following measures:

   (a) $m(\widehat{AB}) = 10$

   (b) $m(\widehat{AC}) = 45$

   (c) $m(\widehat{BD}) = 50$

   (d) $m(\widehat{DF}) = 170$

   (e) What is $m(\widehat{BC})$?

3. (a) How many arc degrees are in a quarter of a circle?

   (b) How many arc degrees are in one-eighth of a circle?

   (c) How many arc degrees are in one-sixth of a circle?

   (d) How many arc degrees are in three-fourths of a circle?

4. Draw a circle with a radius of 2 inches. Starting at any point on the circle use a compass with the same setting (2 inches) to describe a series of equally spaced marks around the circle. CAUTION: Be careful that your compass setting does not change.

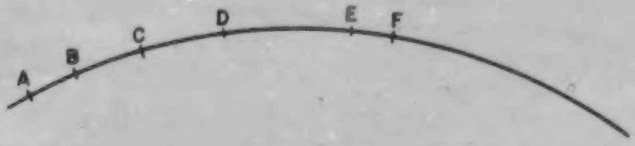
   (a) Do you get back to exactly the same point where you started?

   (b) How many arcs are marked off?

   (c) What is the measure of the central angle of any of these arcs?

   (d) What is the degree measure of any one of these arcs?

5. Refer to the arc $\overset{\frown}{ABCDEF}$, or more briefly $\overset{\frown}{AF}$, shown in the drawing below. Determine the following:



(a) $\overset{\frown}{AC} \cap \overset{\frown}{BD}$

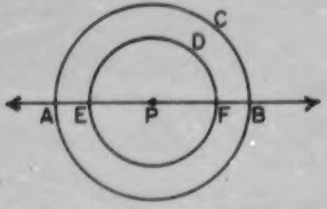
(b) $\overset{\frown}{AF} \cap \overset{\frown}{DF}$

(c) $\overset{\frown}{AD} \cap \overset{\frown}{CF}$

(d) $\overset{\frown}{CD} \cap \overset{\frown}{DE}$

(e) $\overset{\frown}{DF} \cap \overset{\frown}{AE}$

6. In the figure at the right, circle C and circle D are concentric and are in the same plane.



(a) Name a diameter of circle D.

(b) Name a diameter of circle C.

(c) Which circle has the longer diameter?

(d) Which circle seems to be the longer? (The length of a circle is called the circumference.)

(e) Why would it be difficult to measure accurately the length of the circles (the circumference) by following a path around the circles?

7. An experiment to find the relationship between the diameter and the length of a circle.

   (a) Select three circular objects in your home, such as a water glass, a baking pan, a saucer, or a wheel from a toy. (Or you may cut out a circular piece of stiff paper or cardboard.) Use a tape measure (of cloth or flexible steel tape) to find the circumference of each circle. If you do not have a tape measure, use a piece of string, and then measure the length marked off on the string.

   (b) Measure the diameters of the same three circular objects. Since it may be difficult to locate the exact center of the circle, measure across the circle several times to obtain as good a measure of the diameter as possible. The longest of such measurements is the diameter.

   (c) Arrange your results in a table as shown below. Compare the two measures for each object. To compare two quantities, we can find their difference or their ratio. In the table, the "c - d" represents the difference between the measures of the circumference and the diameter. The " $\frac{c}{d}$ " represents the ratio of the measures of the circumference and the diameter. (Find the ratios to the nearest tenth.)

Name of object	Measure of circumference	Measure of diameter	c - d	$\frac{c}{d}$
water glass				
pie pan				
saucer				

   (d) Do the differences in the "c - d" column appear to be the same?

   (e) Do the ratios in the " $\frac{c}{d}$ " column appear to be the same?

[sec. 11-5]

*8. Circle A has a radius of 5 inches. Circle B has a radius of 25 inches. Explain why the arc measure of one-fourth of circle A is the same as the arc measure of one-fourth of circle B.

*9. Demonstrate a one-to-one correspondence between the sets of points on the two semicircles of a given circle which are determined by a diameter.

11-6. Circumference of Circles

It is difficult to measure the circumference of a circle accurately. Remember that the circumference of a circle is the length of the simple closed curve which we call a circle. From the results in Problem 7 in Exercises 11-5, you learned that there seemed to be a relationship between the diameter of a circle and the circumference of that circle.

The difference, "c - d", for one circle does not give us a relation that seems to be true for other circles. For all circles, however, the ratio of the measures of the circumference and the diameter, " $\frac{c}{d}$ " appears to be the same.

What results did you get for the ratio of c to d? Was the number in each case about the same? Was this number a little greater than 3? If your experiment was carefully done, your results for the ratio, $\frac{c}{d}$, should have been about 3.1 or 3.2. It appears that the circumference of any circle is a little more than three times the diameter of the circle.

Mathematicians have proved that, for any circle, the ratio of the measure of the circumference of the circle to the measure of the diameter is always the same number. A special symbol is used for this number. This symbol is written " π ". The symbol is a letter from the Greek alphabet. It is read "pi." π is the first letter in the Greek word for "perimeter." Are the words "perimeter" and "circumference" related in meaning?

In mathematical language, we say that the relation of the measure c of the circumference of a circle to the measure d of its diameter is:

$$\frac{c}{d} = \pi \qquad \text{or} \qquad c = \pi d.$$

The value of π is about 3.14 or $3\frac{1}{7}$. We use this number in finding the circumference of a circle.

It is much easier to measure the diameter of a circle than to measure the circumference. The circumference can be found by using the measure of the diameter and the relation stated above. Assume the diameter of a circle is 5 inches. Then, the length of the circle may be found in the following way:

(Using $\pi \approx 3.14$)

$$c = \pi d$$
$$c = 3.14 \cdot 5$$
$$c = 15.7$$

(Using $\pi \approx 3\frac{1}{7}$)

$$c = \pi d$$
$$c = \frac{22}{7} \cdot 5$$
$$c = 15\frac{5}{7}$$

Exercises 11-6a

1. Find the missing information about the circles described. (Use $\pi \approx 3.14$).

	Circle	Radius	Diameter	Circumference
(a)	A	10 in.	?	?
(b)	B	?	15 ft.	?
(c)	C	3.6 yd.	?	?
(d)	D	4.2 cm.	?	?
(e)	E	?	5.6 in.	?

2. (a) What number is obtained by dividing the measure of the circumference of a circle by the measure of the diameter of the circle?

   (b) How can you determine the circumference of a circle if you know only the diameter?

   (c) Can you determine the diameter of a circle knowing only the circumference? Explain.

3. Copy and complete the following number sentences which show the relation between the circumference of a circle and the diameter of the circle.

   (a) $\pi = \frac{?}{?}$ (b) $? = \pi\ ?$ (c) $\frac{?}{\pi} = ?$

4. Find the missing information about the circle described. (Use $\pi \approx 3\frac{1}{7}$).

	Circle	Radius	Diameter	Circumference
(a)	V	?	?	22 ft.
(b)	W	?	?	25 in.
(c)	X	?	?	16 cm.
(d)	Y	?	?	43 yd.
(e)	Z	?	?	88 mm.

5. (a) How can you determine the measure of the radius of a circle knowing only the measure of the diameter?

   (b) How can you determine the measure of the diameter of a circle knowing only the measure of the radius of the circle?

   (c) How can you determine the measure of the circumference of a circle by using the measure of the radius of the circle?

6. Copy and complete the following number sentences which show the relation between the circumference of a circle and the radius of the circle.

(a) $c = \pi(? \cdot r)$

(b) $c = (\pi \cdot ?)r$

(c) $c = (2\ ?)r$

(d) $\frac{c}{? \cdot ?} = r$

(e) $\frac{c}{r} = ? \cdot ?$

(f) $\frac{c}{2 \cdot r} = ?$

7. Find the missing information about the circles described. (Use $\pi \approx 3.1$).

	Circle	Radius	Circumference
(a)	K	5 in.	?
(b)	L	?	51 ft.
(c)	M	17 cm.	?
(d)	N	?	100 yd.

For Problems 8 through 11, use $\pi \approx 3.14$.

8. A circular lampshade 12 in. in diameter needs new binding around the bottom. How long a strip of binding will be needed? (Disregard any needed for overlap.)

9. A strip of metal 62 inches long is to be made into a circular hoop. What will be its diameter? Would it be large enough for the hoop for a basketball basket? (Official diameter is 18 inches.)

10. A merry-go-round in a playground has a 15-foot radius. If you sit on the edge, how far do you ride when it turns once.

11. A wheel moves a distance of 12 feet along a track when the wheel turns once. What is the diameter of the wheel?

*12. A circle with a diameter of 20 inches is separated by points into 8 arcs of equal length.

(a) What is the length of the whole circle?

(b) What is the length of each arc?

(c) What is the arc measure of each arc?

(d) On this circle, how long is an arc of one degree?

OPTIONAL: The Number π

The number represented by the symbol "π" is a new kind of number. It is not a whole number. Neither is it a rational number. Recall that any decimal expansion of a rational number is a repeating decimal. Mathematicians have proved that π cannot be a repeating decimal. In an article by F. Genuys in Chiffres I (1958), a decimal expression for π to 10,000 decimal places was published.

Here is the decimal for π to fifty-five places.
3.14159 26534 89793 23846 26433 83279 50288 41971 69399 36510 58209...

(The three dots at the end indicate that the decimal expression continues indefinitely.)

If you examine this decimal for the number π you see that it is difficult to locate the point on the number line to which it corresponds. However, by examining the digits in order we can find smaller and smaller segments on the number line which contain the point for π.

Study the following statements:

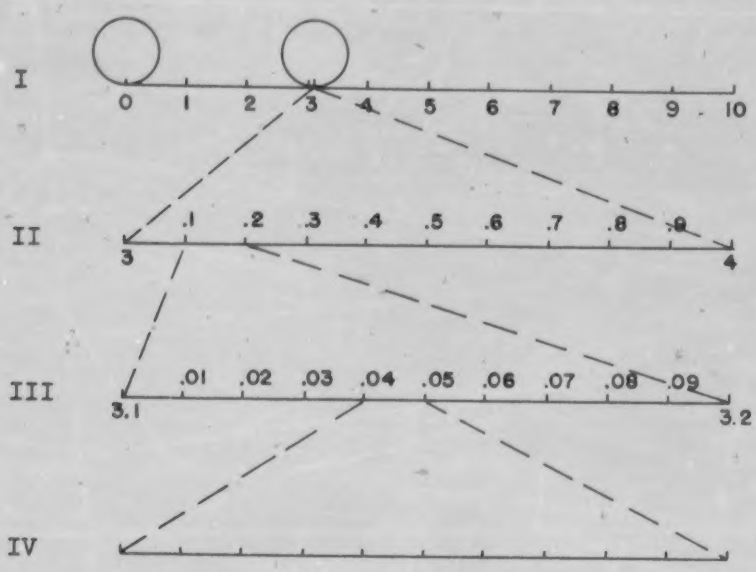
1. $\pi = 3 + .141592...$ Therefore $\pi > 3$, and $\pi < 4$.
Why? $(3 < \pi < 4.)$

2. $\pi = 3.1 + .041592...$ Therefore $\pi > 3.1$, and $\pi < 3.2$.
(Why?) $(3.1 < \pi < 3.2)$

3. $\pi = 3.14 + .001592...$ Therefore $\pi > 3.14$, and $\pi < 3.15$.
(Why?) $(3.14 < \pi < 3.15)$

What should the next statement be?

In the figure below is shown a number line. A circle with diameter one unit in length is shown in its first position at point 0, and also in its position after rolling along the number line. The approximate position of the point which corresponds to the number π is the point of tangency, C.



Line I in the figure illustrates Statement 1. The point corresponding to π is on the segment with endpoints 3 and 4.

The segment on line I with endpoints 3 and 4 is shown enlarged (ten times as large) on line II. The segment is subdivided to show tenths, so the points of division correspond to the numbers 3.1, 3.2, 3.3, etc. Statement 2 tells us that the point for π is on the segment with endpoints 3.1 and 3.2.

The segment with endpoints 3.1 and 3.2 is shown enlarged (ten times as large) on line III. The points on line III subdivide it into tenths, and correspond to the numbers 3.11, 3.12, etc. From Statement 3 we know that the point corresponding to π is on the segment with endpoints 3.14 and 3.15.

What numbers should be at the endpoints of the segment marked IV? How should the points of subdivision be labeled? On what segment of line IV is the point for π ?

Look again at line III. Notice that the whole segment on line III was made 100 times as large as it would actually be on the number line I, on which the circle was rolled. Using just three digits in the decimal for π we have shown that the point π lies on a particular segment which is very small--just $\frac{1}{100}$ of the segment of the number line between the points 3 and 4.

If you used one more digit, you could show that the endpoints of a segment which contains π are __?__ and __?__.

Exercises 11-6b.

1. (a) Find, to three decimal places, the difference between π and each of the following rational numbers:

   $\frac{19}{6}$ $\frac{22}{7}$ $\frac{25}{8}$

   (b) Which of the rational numbers in Problem 2 is nearest π? Using $\pi \approx \frac{22}{7}$, find the following:

   (c) The length of a circle whose diameter is 14 in.

   (d) The length of a circle with radius 21 ft.

   (e) The diameter of a circle with circumference 132 in.

   (f) The radius of a circle with circumference 44 ft.

   (g) The circumference of a circle with radius $10\frac{1}{2}$ in.

2. (a) Find, to four decimal places, the decimal for 2π.

   (b) Find, to five decimal places, the decimal for 3π.

3. Sometimes it is a good idea to use π as a numeral, instead of using a decimal for π. Answer the following questions using π as a numeral. We say the answer is expressed "in terms of π".

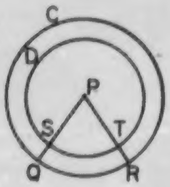
   (a) If the length of a circle is 54π in., what is its diameter? Its radius?

   (b) If the diameter of a circle is 13 in., what is the length of the circle?

   (c) If the radius of a circle is 3.6 cm., what is the length of the circle?

4. Suppose that the diameter of circle C is three times as long as the diameter of circle D. What is the ratio of measures of their circumferences? (Hint: Think of lengths for the diameters, and then find the circumferences. Use π as a numeral.)

5. In the figure, circle C and circle D have the same center P. The radius of circle C is 7 in. and the radius of circle D is 5 in.



   (a) Find the length of each circle.

   (b) If angle QRP contains 70 degrees, what is the arc measure of arc $\overset{\frown}{ST}$? Of arc $\overset{\frown}{QR}$?

   (c) What fractional part of circle C is arc $\overset{\frown}{QR}$? What fractional part of circle D is $\overset{\frown}{ST}$?

   (d) What is the linear measure of arc $\overset{\frown}{QR}$? Of arc $\overset{\frown}{ST}$?

6. BRAINBUSTER. Suppose a tight band is placed around the earth, over the equator. Then suppose the band is made 1 foot longer which loosens the band, and that the band is the same distance from the earth all the way around.

   (a) Could a mouse crawl under the band?

   (b) Could you crawl under the band? If not, how much longer would the band have to be to make it possible for you to crawl under it?

   (c) What mathematical property could you use to answer these questions?

11-7. Area of a Circle

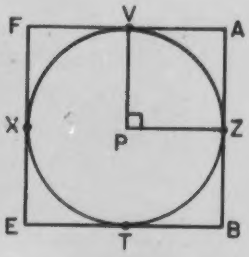
In the kitchen at home you may find a circular frying pan called a nine-inch skillet. The boundary of the frying surface represents a circle, and "nine-inch" tells the diameter of the circle. Some skillets are square shaped. Perhaps you can find an eight-inch square frying pan too. A person trying to decide whether to buy a nine-inch circular pan or an eight-inch square pan might ask himself, "Which one is bigger?" By "bigger," we mean more surface usable in cooking. In other words, one might want to compare the area of the closed region of an eight-inch square with the area of the closed region of a nine-inch circle.

After a careful inspection of the two skillets, you might conclude that the areas are so nearly the same that you cannot decide which is greater. Let us try to find out which skillet has the greater surface for cooking.

When speaking about a circle in everyday language, we usually use the phrase "the area of a circle" when we mean "the area of the closed circular region." After we learn how to compute the area of a circle, we can express the area of a circle in terms of its radius. This is frequently done when talking about a circular region.

Since each side of the square is eight inches long, we know its area is sixty-four square inches. But, we have not yet studied a method of computing the area of a circle. We can measure the diameter. This measure can be used to compute the radius or the circumference. But it is difficult to measure the area of a circle.

In the figure, the point P is the center of the circle and also the center of the square ABEF. Let the measure of a radius of the circle be r. Then, each of the segments $\overline{VP}$ and $\overline{PZ}$ is a radius with a measure r. Angle VPZ is a right angle. Does the square ABEF have four times as much area as the square VAZP?



Now answer the following questions: (Note that in the drawing the circle has a diameter which is equal in length to an edge of the square.)

1. (a) Is the area of the circle greater or less than the area of square ABEF?

   (b) Is the area of the circle greater than the area of square VAZP?

   (c) Is the area of the circle greater or less than four times the area of the square VAZP?

   (d) Does this tell us the area of the circle?

Let us try approximating the area of a circle by a rather careful measurement. On the next page, at the top, is shown a small square. Each of the sides of the small square is one unit long. The small square represents one unit of area.

The large figure shows a circle whose radius is ten units in length. The large square, whose sides are tangent to the circle, is twenty units long. The region enclosed by the circle has been covered with units of area. Use this drawing to find the approximate area of the circle.

What would be an efficient way to count the number of square units in the circular region? We could count the number of such units in the quarter-circle bounded by the square ABCD. Then we could multiply this number by 4. Note, however, that some of the units of area are partially in the exterior of the circle and partially in the interior. We must estimate the total number of square units. This can be done by counting those units which are more than half in the interior as a whole unit. Then we disregard the units for which the smaller part is in the interior.

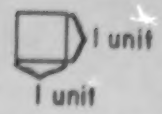
There are about 79 units of area in the quarter-circle. The number of units of area in the entire circle is therefore about 4 times 79, or 316.

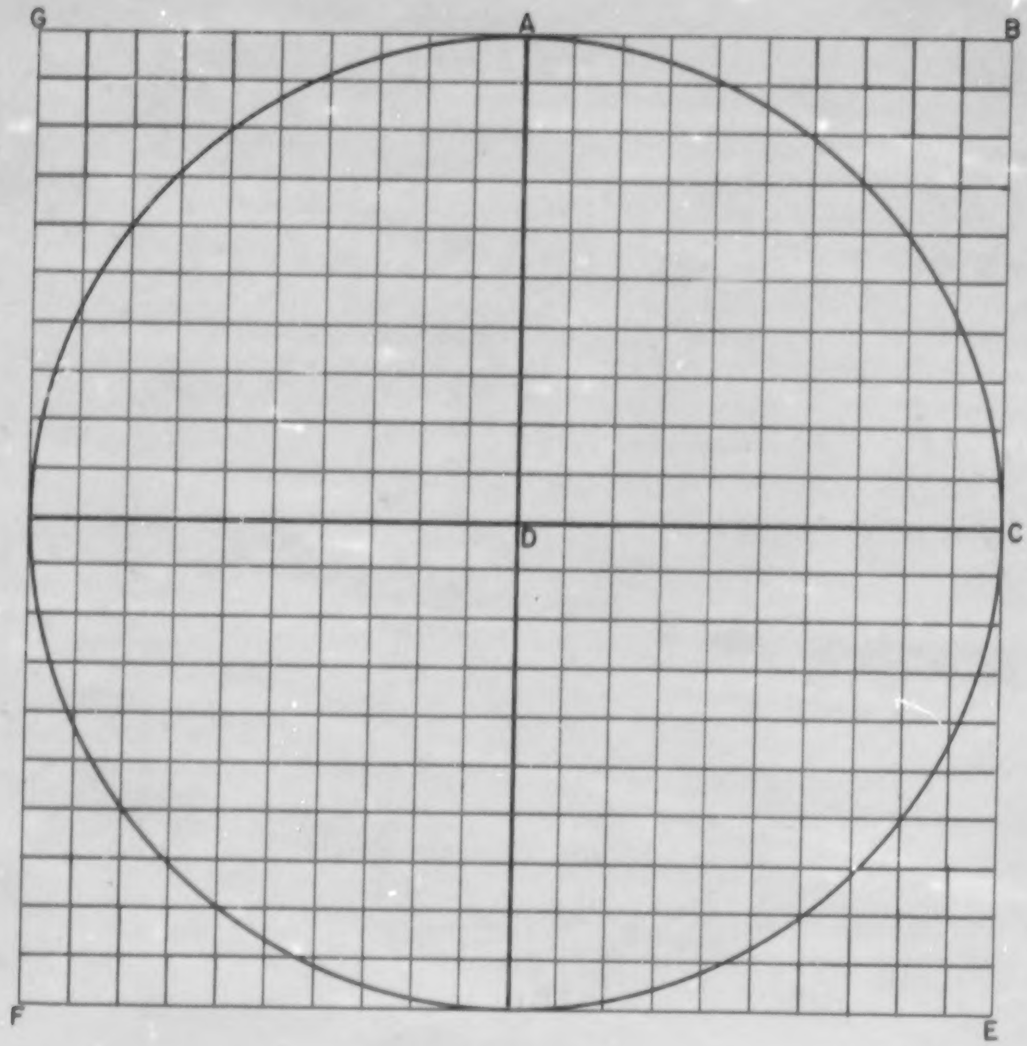
What is the area of the square ABCD? The area of ABCD is 100 square units. Then the area of BEFG is 400 square units. Now answer the following questions:

II. (a) How does the area of the circle compare with the area of BEFG?

(b) Is the area of the circle a little more than three times the area of ABCD?

(c) How can you quickly determine the area of the square ABCD?


1 unit
1 unit


G
A
B
C
D
F
E

As we have seen from the answers to the above questions, the area of BEFG is greater than the area of the circle. Since the area of BEFG is four times the area of ABCD, then, four times ten times ten $(4 \cdot 10 \cdot 10)$ results in a product greater than the area of the circle. But, the area of the circle is a little more than three times ten times ten $(3 \cdot 10 \cdot 10)$.

Mathematicians have proved that for any circle, the area of the circle is a little more than three times the measure of the radius multiplied by itself. (The measure of the radius of the circle in the drawing was 10. This was the same as the measure of the length of the square ABCD.) The area of a circle is equal to the product of π and the square of the radius. In mathematical language, we say,

$$A = \pi r^2,$$

where A is the number of units of area and r is the measure of radius.

Let us return to the comparison of the two skillets. A nine-inch circular skillet has a radius of $\frac{9}{2}$ inches (or $4\frac{1}{2}$ in.). The area of the skillet may be found as follows:

$$A = \pi r^2 \quad (\text{Using } \pi \approx 3\tfrac{1}{7})$$

$$A \approx \frac{22}{7} \cdot \frac{9}{2} \cdot \frac{9}{2}$$

$$A \approx \frac{1782}{28} \text{ or } 63\tfrac{18}{28}$$

Now we can answer the question, "Which has the greater surface for cooking, a nine-inch circular skillet or an eight-inch square skillet?" What is your answer?

Exercises 11-7.

1. Find the area of each circle for which the radius is given. (Use $\pi \approx 3\frac{1}{7}$)

(a) 7 in.
(b) 5 ft.
(c) 14 cm.
(d) 21 yd.
(e) 3.5 mm.
(f) 4.2 yd.

2. Find the area of each circle for which the radius is given. (Use $\pi \approx 3.14$)

(a) 8 ft.
(b) 10 yd.
(c) 15 cm.
(d) 20 ft.
(e) 18 in.
(f) 2.5 yd.

3. Information is given for five circles. The letters r, d, c, and A are the measures of the radius, the diameter, the circumference, and the area respectively. Find all the missing data. Use 3.1 as an approximation to π.

	Circle	Radius	Diameter	Circumference	Area
(a)	A	4 ft.			
(b)	B		16 cm.		
(c)	C		20 ft.		
(d)	D			100 mi.	
(e)	E	111 in.			

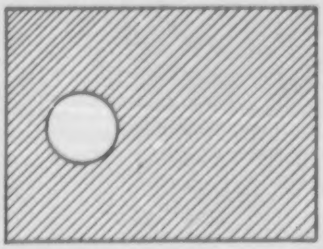
4. Which has the greater frying surface--an eight-inch circular skillet or a seven-inch square frying-pan? ($\pi \approx 3.14$)

5. A circular drumhead is twelve inches across. What is the area of the drumhead?

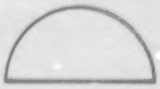
6. (a) Which method is easier for finding the area of a circle--to measure the radius and calculate the area, with the aid of πr^2, or to measure the area directly with an appropriate unit of measure?

   (b) Should both methods yield the same result?

7. A rectangular plot of land, 40 feet by 30 feet, is mostly lawn. The circular flower-bed has a radius of 7 feet. What is the area of the portion of the plot that is planted in grass?



8. The figure represents a simple closed curve composed of an arc of a circle and a diameter of the circle. The area of the interior of this simple closed curve, measured in square inches, is 8π Do not use any approximation for π in this problem.



   (a) What is the area of the interior of the entire circle?

   (b) What is the second power of the radius of the circle?

   (c) How long is a radius of the circle?

   (d) How long is the straight portion of the closed curve represented in the figure?

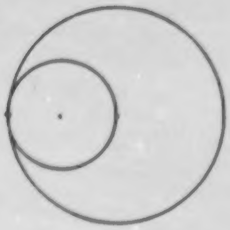
   (e) What is the circumference of the (entire) circle?

   (f) How long is the circular arc represented in the figure?

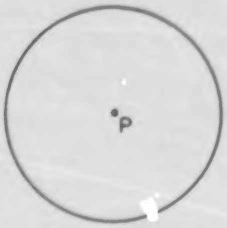
   (g) What is the total length of the simple closed curve?

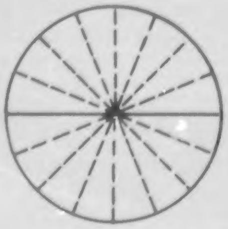
*9. The earth is about 150 million kilometers from the sun. The orbit (or path) of the earth around the sun is not really circular, but approximately so. Suppose that the orbit were a circle; then the path would lie in a plane and there would be an interior of the orbit (in the plane). What would be an estimate for the area of this interior?

*10. The center of the longer circle lies on the shorter circle. The intersection of the two circles is a single point. This point and the centers of the two circles lie on one line. If the interior of the shorter circle is chosen as a unit of measure, what would be the measure of the region inside the longer circle and outside the shorter circle?

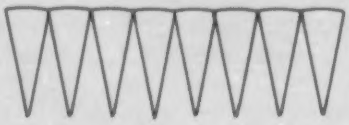


*11. Another way to discover a relationship between the radius of a circle and the area of the same circle is the following. Perform the suggested steps. On a piece of stiff paper draw a large circle with your compass. Mark the center P of the circle. (See left-hand figure.) Use your protractor to assist you in drawing eight lines on P which will separate the interior of the circle into sixteen regions all of the same area. (See right-hand figure.) Can you calculate how many arc-degrees each of the sixteen arcs will have?

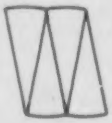




Cut away the portion of the paper outside the circle. Cut through the interior of the circle along the fully-drawn diameter. In each of the two halves, cut with great care along the dotted radii from P almost to the circle itself. The eight angular portions should hang together somewhat like teeth.



With both portions cut in a toothed fashion, fit the two pieces together. (Only a few of the sixteen teeth are shown in this figure.)



The upper and lower boundaries of the completed pattern have a scalloped appearance. If they were straight, the entire figure would be the interior of a ____________ (fill the blank with the best choice of a name for this simple closed curve). As an application of results from Chapter 10 you may estimate the area of the interior of this curve by using the measures of its apparent base and its apparent altitude. How does your result compare with the product of π and the second power of the radius?

12. BRAINBUSTER. A barn is 40 feet long and 20 feet wide. A chain 35 feet long is attached to the barn at the middle of one of its longer sides. Another chain 35 feet long is attached to the barn at one of its corners. Either chain may be used to tether a cow for grazing.

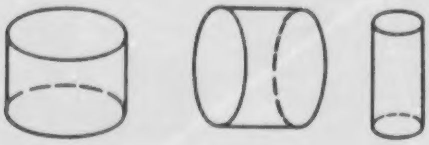
    (a) Which chain gives the tethered cow the greater area of land over which to graze?

    (b) How much difference is there between the areas of the two regions? (Use 3.1416 as an approximation to π.)

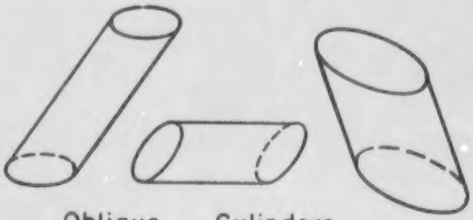
11-8. Cylindrical Solids--Volume

In Chapter 8, you studied the rectangular solid, its volume and its surface area. In Chapter 10, you studied the prism, its volume, and its surface area. Here we shall study another solid which is frequently found in everyday life. Instead of having a rectangular base, like a box, suppose a solid has a circular base, like a tin can. We call such a solid a *cylindrical solid* (or sometimes just a *cylinder*). You are familiar with other examples of cylindrical solids such as water pipes, tanks, silos, and some drinking glasses.

The figures shown below represent cylinders. Those on the left are called *right cylinders*. Compare them with the *oblique* (or slanted) *cylinders* on the right.



Right Cylinders



Oblique Cylinders

We rarely see slanted cylindrical solids in ordinary life. Therefore in this chapter we shall assume our solids are right cylindrical solids.

We list some important properties of a right cylinder.

(1) It has two congruent bases (a top and a bottom) and each is a circular region.

(2) Each base is in a plane and the planes of the two bases are parallel.

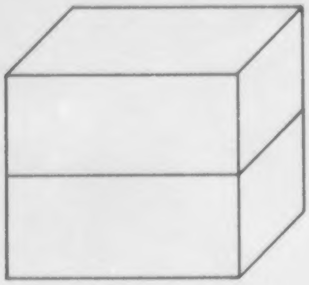
(3) If the planes of the bases are regarded as horizontal, then the upper base is directly above the lower base.

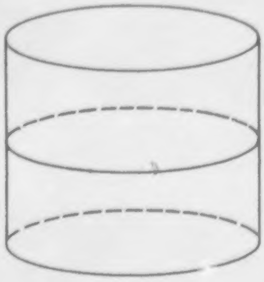
(4) The lateral or side surface of the cylinder is made up of the points of segments each joining a point of the lower circle with the point directly above it in the upper circle.

There are two numbers or lengths which describe a cylindrical solid. These are the radius of the base of the cylinder and the altitude (or height) of the cylinder. The altitude is the (perpendicular) distance between the parallel planes containing the bases. For a right cylinder, the altitude can also be thought of as the length of the shortest possible segments lying in the lateral surface and joining the two bases.

How can we find the volume of a cylindrical solid? In one sense there is a fairly easy method. If the solid is like a tin can and will hold water (or sand) we can fill it up and then pour it into a standard container. However, we would like to know what the answer is without having to do this every time. For some cylinders, this method would be very impractical, perhaps impossible.

Recall how we found the volume of a box or of a (right) prism. We first considered a box or prism one unit high. The number of cubic units in this box or prism would be the same as the number of square units in the base. Thus the measure of the volume was clearly the measure of the area of the base times one. If the box or prism had an altitude of two units, then the measure of the volume would clearly be twice as much as the measure of the area of the base. That is, it would be 2 times the measure of the area of the base.





In general, if the area of the base were B square units and the altitude of the box or prism were h units then the volume would be $B \cdot h$ cubic units.

Exactly the same situation occurs with a cylindrical solid. The measure of the volume of the cylinder is simply the measure of the area of the base times the measure of the altitude. The area of the base of a cylinder is πr^2 square units. So the volume is $\pi r^2 \cdot h$ cubic units.

We now have one basic principle which applies to boxes, to other prisms, and to cylinders. The measure of the volume is the measure of the area of the base times the measure of the altitude. In mathematical terms, this is frequently written as follows:

$$V = Bh$$

where B represents the measure of the area of the base and h represents the measure of the height.

You should learn and remember how to compute the volume of a cylindrical solid. To compute the volume of any solid of this type we simply multiply the measure of the area of the base by the measure of the altitude. The altitude is the (perpendicular) distance between the parallel planes which contain the bases. If you think of the geometrical figure and what it is you want to find, then most problems of this type are very easy.

As an example, assume you want to find the volume of a cylinder whose radius has a measure of 7 and whose height (or altitude) has a measure of 10.

$$V = Bh; \text{ for a cylinder, } B = \pi r^2$$

Therefore, $V = \pi r^2 h$

$$V = \frac{22}{7} \cdot 7 \cdot 7 \cdot 10$$

$$V = 1540$$

The volume of the cylinder is about 1540 cubic units.

A note on computation. Sometimes when making computations involving π, it is usually easier to use a decimal approximation for π only at the last step of the arithmetic. In this way we use long decimals as little as possible. Consider $\pi \cdot 5^2 \cdot 8$. We note that $5^2 = 25$ and that $25 \cdot 8 = 200$. Therefore $\pi \cdot 5^2 \cdot 8 = \pi \cdot 200 \approx (3.14) \cdot (200) = 628$. This procedure is much simpler than multiplying 3.14 by 25 and then multiplying the result by 8.

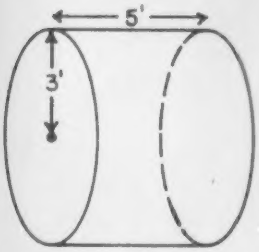
Exercises 11-8

1. Information is given for five right cylinders. The letters r and h are the measures of the radius of the circular base and the height of the cylinder respectively. Using 3.1 as an approximation for π, find the volumes of each cylinder.

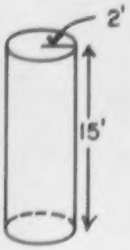
	Cylinder	Radius (r)	Height (h)	Volume
(a)	A	4 in.	8 in.	?
(b)	B	8 ft.	4 ft.	?
(c)	C	10 cm.	30 cm.	?
(d)	D	7 yd.	25 yd.	?
(e)	E	12 in.	12 in.	?

2. Find the volumes of the right cylinders shown here. The dimensions given are the radius and the height of each cylinder. The figures are not drawn to scale. (Use $\pi \approx 3.1$).

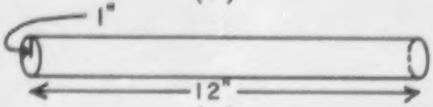
(a)



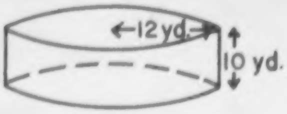
(b)



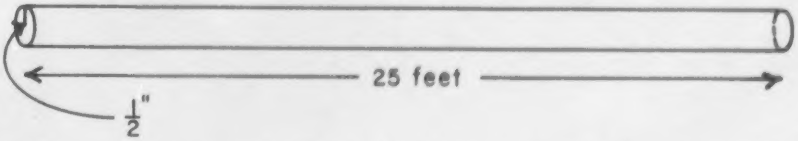
(c)



(d)



(e)



3. A silo (with a flat top) is 30 feet high and the inside radius is 6 feet. How many cubic feet of grain will it hold? (What is its volume?) Use $\pi \approx 3.14$.

4. A cylindrical water tank is 8 feet high. The diameter (not the radius) of its base is 1 foot. Find the volume (in cubic feet) of water which it can hold. Leave your answer in terms of π. If you use an approximation for π, what is your answer to the nearest (whole) cubic foot?

5. There are about $7\frac{1}{2}$ gallons in a cubic foot of water. About how many gallons will the tank of Problem 2 hold?

6. Find the amount of water (volume in cubic inches) which a 100 foot length of pipe will hold if the inside radius of a cross-section is 1 inch. Use $\pi \approx 3.14$. (A cross-section is shaped like the base. A cross-section is the intersection of the solid and of a plane parallel to the planes of the bases and between them.)

*7. Find the volume of a cylindrical solid whose altitude is 10 centimeters and the radius of whose base is 3 centimeters. Leave your answer in terms of π.

*8. In Problem 7, what would the volume be if the altitude were doubled (and the base were left unchanged)?

*9. In Problem 7, what would be the volume if the radius of the base were doubled (and the altitude were left unchanged)?

*10. In Problem 7, what would be the volume if the altitude were doubled and the radius of the base were also doubled? (Think of first doubling the altitude and then doubling the radius of this new cylindrical solid.)

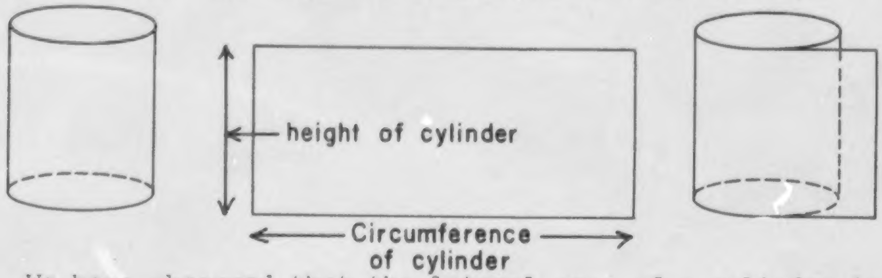
*11. In general, what is the effect on the volume of a cylindrical solid obtained by doubling the altitude? By doubling the radius of the base? By doubling both the radius of the base and the altitude?

12. BRAINBUSTER. What is the amount (volume) of metal in a piece of water pipe 30" long if the inside diameter of a cross-section is 2" and the outside is 2.5". Use $\pi \approx 3.1$.

11-9. Cylindrical Solids--Surface Area

In the previous section we considered questions about the volume of a cylinder. We will now consider the surface area. There are two questions that might be asked: (1) What is the area of the curved surface? (2) What is the total surface area? It is easy to see the relation between these two if we think of the solid itself. The total area is the area of the curved surface plus the area of the top base plus the area of the bottom base. But the areas of the top and bottom bases are the same. And the measure of each is πr^2 where r is the measure of the radius of the base. So if we know how to find the area of the curved surface, we can find the total area.

The label of a tin can almost covers the curved surface of the can. We will assume the label covers the entire curved surface of a can. Then, the area of the label is the lateral area of the cylinder. How are labels made? They are made and printed in the form of rectangular regions. The height of such a rectangle is the height of the cylindrical solid. The length of the base is the circumference of the base circle of the cylinder. (When made, the label has this length plus a little more, to allow for overlapping.) The lateral area of a cylinder, then, is merely the area of a rectangle as shown below.



We have observed that the lateral area of a cylinder is the area of a certain rectangle. The altitude of the rectangle and the altitude of the cylinder are the same. The length of the base of the rectangle and the circumference of the base of the cylinder are the same. Therefore the measure of the lateral area of the cylinder is the product of the measure of the length of the base circle and the measure of the height. Hence, the measure is

$$2\pi r \cdot h.$$

And the measure of the total area is then

$$2\pi r \cdot h + 2\pi r^2.$$

There are some curved surfaces, like the surface of a ball, which cannot be treated in quite this simple a way. Rectangular regions, or other flat surfaces, just don't wrap nicely around them. The areas of such surfaces can be handled in other ways. It is fortunate that cylinders have "easy" curved surfaces.

Two formulas presented in this chapter are very important in dealing with measurements of circles. These two formulas are:

(1) $c = 2\pi r$ (or $c = \pi d$)

(2) $A = \pi r^2$

The formulas for the volume and surface area of a cylindrical solid use these two formulas. Therefore it is not essential that you memorize the formulas for volume and surface area, if you can remember what the formulas represent.

You should understand and know the basic general principle for getting the volumes of many solids. For solids, which are right prisms and right cylinders, this principle tells us that the measure of the volume is the product of the measure of the area of the base and the measure of the altitude, or in mathematical symbols, $V = Bh$.

Finally, in many problems such as finding surface areas, you should think of the geometric objects and what you want to know. As an example, in finding the lateral surface of a cylindrical solid think of what the lateral surface represents if flattened out. Then the lateral area is the area of a rectangle. To get the total surface area, add to the lateral area twice the area of the base. In mathematical symbols,

$$S_T = \pi dh + 2\pi r^2$$

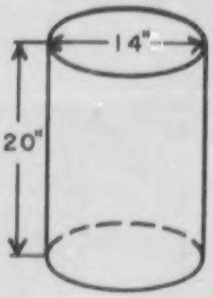
where S_T represents the measure of the total surface of a cylinder,

d represents the measure of the diameter of the base,

r represents the measure of the radius of the base.

h represents the measure of the height of the cylinder.

To find the total surface area of the cylinder shown at the right: (Using $\pi \approx 3\frac{1}{7}$).



$$S_T = \pi dh + 2\pi r^2$$

$$S_T = \left(\frac{22}{7} \cdot 14 \cdot 20\right) + \left(2 \cdot \frac{22}{7} \cdot 7 \cdot 7\right)$$

$$S_T = 880 + 308$$

$$S_T = 1188$$

The total surface area is about 1188 square inches.

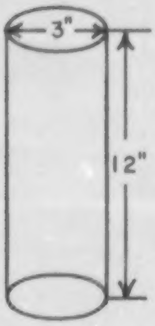
Exercises 11-9

1. Information is given for five cylinders. Using $\pi \approx 3.1$, find all the missing data.

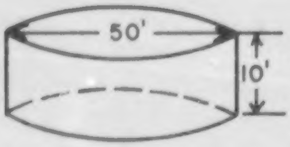
	Cylinder	Radius of base	Diameter of base	Height	Total Surface Area (A_t)
(a)	A	?	10 in.	10 in.	?
(b)	B	1 ft.	?	3 ft.	?
(c)	C	?	16 ft.	17 ft.	?
(d)	D	15 cm.	?	50 cm.	?
(e)	E	?	8 yd.	12 yd.	?

2. Find the total surface area for each of the cylinders shown. (Use $\pi \approx 3.1$).

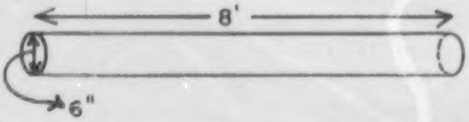
(a)



(b)



(c)



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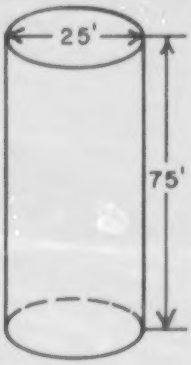


3. HOME PROJECT: Complete the following project at home.

   (a) Take an ordinary tin can. Measure it and make a label for it that will fit without overlapping. Your label will be in the form of a _______. Its height will be the ________ of the tin can. The length of its base will be the __________ of the tin can. Try your label to see that its fits.

   (b) Take another tin can of different size from that of (a). Make a label for it without doing any measuring. Determine the desired size of the label by comparison with the tin can.

   (c) With a tape measure or string (to be measured) figure out the lateral surface area of the tin can of (b) by measuring the circumference of the base and the height directly.

   (d) Measure the diameter (and then figure out the radius) of the base of the tin can of (b). Using this and the height find the lateral surface area of the tin can. Should your answer agree with the answer you got in (c)?

4. Find the lateral surface area (in square centimeters) of a cylindrical solid whose altitude is 8 centimeters and the radius of whose base is $1\frac{1}{2}$ centimeters. Use $\pi \approx 3.14$.

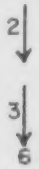
5. Find the total surface area of the cylinder of Problem 4.

*6. How many square meters of sheet metal do you need to make a closed cylindrical tank whose height is 1.2 meters and the radius of whose base is .8 meters? How many square meters do you need if the tank is to be open on top?

*7. A small town had a large cylindrical water tank that needed painting. A gallon of paint covers about 400 square feet. How much paint is needed to cover the whole tank if the radius of the base is 8 feet and the height of the tank is 20 feet? Give your answer to the nearest tenth of a gallon.

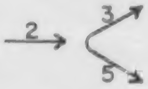
11-10. OPTIONAL: Review of Chapters 10 and 11.

1. It is often helpful to use diagrams to show relations. For example



may represent $2 \cdot 3 = 6$

Relations which involve sums as well as products may be shown by diagrams.



may represent

$$2(3 + 5) = 2 \cdot 3 + 2 \cdot 5$$
$$= 6 + 10$$

You have studied ways of computing perimeters, areas, and volumes for a number of geometric figures and their interiors; and you have expressed these methods briefly in number sentences or formulas.

Below are twelve diagrams for several of these formulas, and a list of the geometric figures to which one (or more) of the formulas applies. Copy each diagram on your paper, and beside it write the following:

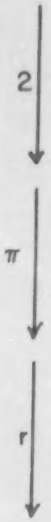
(a) The name of the figure (or figures) to which the diagram applies. (Choose from the list of figures below.)

(b) What the formula tells about the figure: area, perimeter, volume, circumference, lateral area.

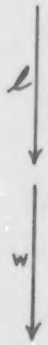
(c) The formula shown by the diagram.

List of figures:
- parallelogram
- rectangular prism
- circle
- triangle
- rectangle
- triangular prism
- cylinder

(1)



(2)



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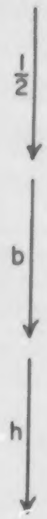

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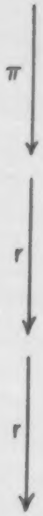
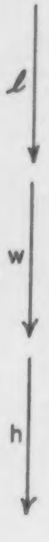
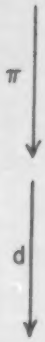


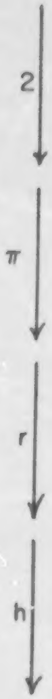
(5)



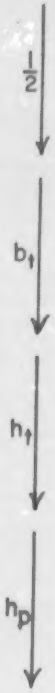
(6)



(7)



(8)



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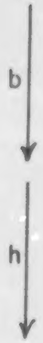

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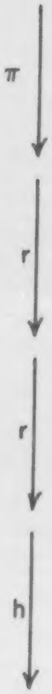

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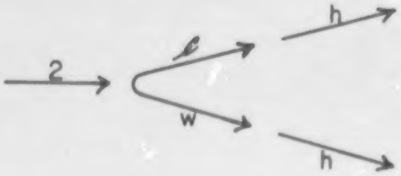

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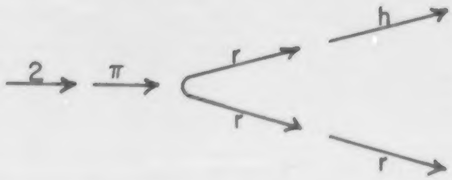


2. (a) In your copies of the diagrams, draw a circle around each numeral.

(b) Draw a square around each symbol for a measure you would have to know in order to apply the formula to a particular figure.

(c) Hold a ruler horizontally across the diagrams (1) to (10) in your text, to separate the numerals from the symbols for measures. Why are there more arrows below your ruler in some diagrams than in others?

(d) Each of the formulas tells how to find the measure of a quantity which has either one dimension, two dimensions, or three dimensions. Put a "1", "2", or "3" beside each diagram, to show the dimension of the kind of quantity to which it relates.

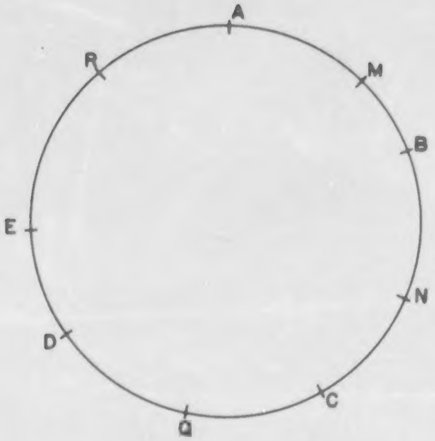
(e) Do you see any connection between your answers for Questions 2(b) and 2(d)?

(f) Look at the diagrams of formulas for finding volumes. (There are three.) Put a B at the end of one arrow, to separate the part of the formula which gives the area of the base from the rest of the formula.

*(g) Make up formulas relating to squares and cubes. Make diagrams for the formulas, and tell what each diagram means.

*(h) Draw the diagrams (11) and (12) in a different way. Write the formulas for your new diagrams.

3.



In the above figure determine with the use of your protractor the measure of the following arcs. Indicate your results with correct use of symbols.

(a) $\overset{\frown}{AMB}$

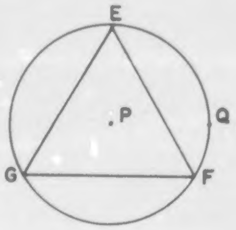
(b) $\overset{\frown}{ABC}$

(c) $\overset{\frown}{CQD}$

(d) $\overset{\frown}{CDE}$

(e) $\overset{\frown}{RAM}$

4. Each side of triangle EFG is measured as 34.6 meters long. The distance between E and the center P of the circle is measured as 20.0 meters. The altitude of triangle EFG from E to the side $\overline{FG}$ is measured as 30.0 meters long.



(a) What is the area of the triangle EFG.

(b) What is the area of the circle? (Use 3.142 as an approximation to π, and round your answer to the nearest squaremeter.)

(c) On your paper draw a sketch of the figure and shade the intersection of the interior of the circle and the exterior of the triangle EFG.

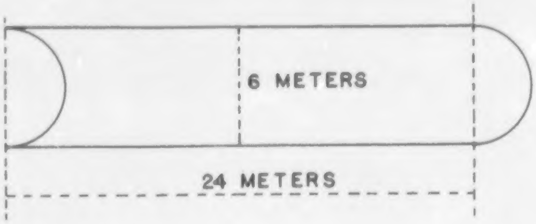
(d) What is the area of this intersection?

(e) The half-plane on the Q-side of EF and the interior of the circle have an intersection. What is the area of this intersection?

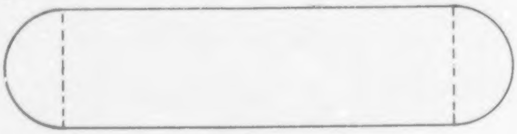
5. Below are four figures representing simple closed curves. Each curve is the union of several segments and one or two arcs of circles. Each arc either is a semicircle or is measured as 90 arc-degrees. The dotted segments are not parts of the simple closed curves but are useful in indicating the lengths.
For each curve find its total length and find the area of its closed region.

(a)

(Use $\pi \approx 3$.)

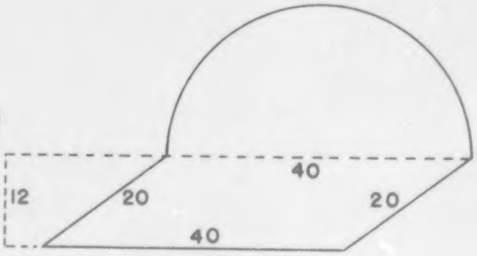


(b) Each segment is 31.4 centimeters long. The distance between the parallel segments is 17.9 centimeters. (Use $\pi \approx 3.14$, and round final answers to nearest centimeter or nearest square centimeter.)



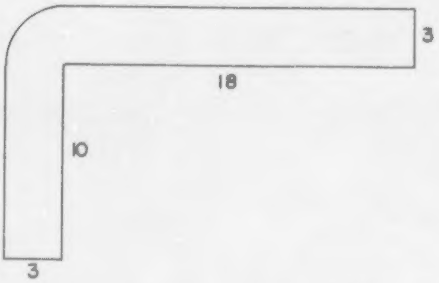
(c)

Unit of measurement is foot. (Leave your answer in terms of π.)

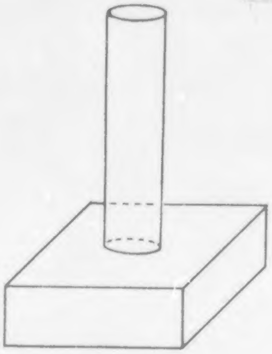


*(d)

Unit of measurement is millimeter. (Leave your answer in terms of π.)



6. Find the volume of a momument made as follows: The base is a rectangular block of marble of dimensions 4' by 6' by 2' high. On top of this block in the center is a cylindrical solid of height 8' and with the radius of the base circle 1'. Use $\pi \approx 3.1$.

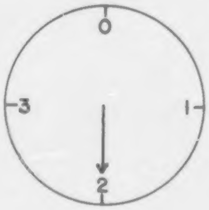


*7. Find the total area of the exposed surface of the monument of the preceding problem. The underside of the base is considered as not exposed.

Chapter 12

MATHEMATICAL SYSTEMS

12-1. A New Kind of Addition.



The sketch above represents the face of a four-minute clock. Zero is the starting point and, also, the end-point of a rotation of the hand.

With the model we might start at 0 and move to a certain position (numeral) and then move on to another position just like the moving hand of a clock. For example, we may start with 0 and move $\frac{2}{4}$ of the distance around the face. We would stop at 2. If we follow this by a $\frac{1}{4}$ rotation (moving like the hand of a clock), we would stop at 3. After a rotation of $\frac{2}{4}$ from 0 we could follow with a $\frac{3}{4}$ rotation. This would bring us to 1. The first example could be written $2 + 1$ gives 3 where the 2 indicates $\frac{2}{4}$ of a rotation from 0, the + means to follow this by another rotation (like the hand of a clock), and the 1 means $\frac{1}{4}$ rotation, thus we arrived at the position marked 3 (or $\frac{3}{4}$ of a rotation from 0). The second example would be $2 + 3$ gives 1 where the 2 and + still mean the same as in the first example and the 3 means a rotation of $\frac{3}{4}$. A common way to write this is:

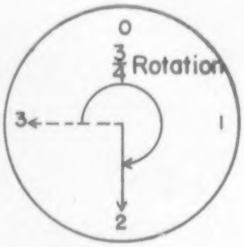
$$2 + 3 \equiv 1 \pmod 4$$

which is read:

Two plus three is equivalent to one (mod 4).

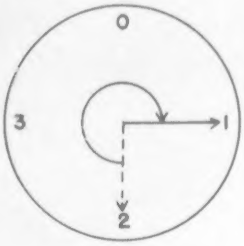
The (mod 4) means that there are four numerals: 0, 1, 2, 3 on the face of the clock. The + sign means what we described above - this is our new type of addition. The ≡ between the 2 + 3 and the 1 indicates that 2 + 3 and 1 are the same (that is, "equivalent") on this clock. We call this briefly "addition (mod 4)." Of course there are other possible notations which could be used but this is the usual one. The expression "(mod 4)" is derived from the fact that sometimes 4 is called "the modulus" which indicates how many single steps are taken before repeating the pattern.

Example 1. Find 3 + 3 (mod 4).

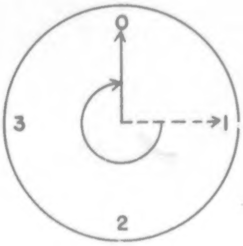


$3 + 3 \equiv 2 \pmod 4$

Example 2. Find (2 + 3) + 3 (mod 4).



$2 + 3 \equiv 1 \pmod 4$



$1 + 3 \equiv 0 \pmod 4$

$(2 + 3) + 3 \equiv 1 + 3 \equiv 0 \pmod 4$

The following table illustrates some of the addition facts in the (mod 4) system.

(Mod 4)

+	0	1	2	↓ 3
0	0	1	2	
1			3	0
→ 2				1
3		0		

We read a table of this sort by following across horizontally from any entry in the left column, for instance 2, to the position below some entry in the top row, such as 3 (see arrows). The entry in this position in the table is then taken as the result of combining the element in the left column with the element in the top row (in that order). In the case above we write $2 + 3 \equiv 1 \pmod{4}$. Use the table to check that $3 + 1 \equiv 0 \pmod{4}$.

Example 3. Complete the following number sentences to make them true statements.

(a) $3 + 4 \equiv ? \pmod{5}$

The mod 5 system represented by the face of a clock should have five positions; namely, 0, 1, 2, 3, and 4. If you draw this clock you will see that $3 + 4 \equiv 2 \pmod{5}$ since the 3 means a rotation of $\frac{3}{5}$ from 0. This is followed by a $\frac{4}{5}$ rotation which ends at 2.

(b) $2 + 3 \equiv ? \pmod{5}$

$2 + 3 \equiv 0 \pmod{5}$. This is a $\frac{2}{5}$ rotation from 0 followed by a $\frac{3}{5}$ rotation which brings us to 0.

(c) $4 + 3 \equiv ? \pmod{6}$

In the mod 6 system, the positions on the face of the clock are marked 0, 1, 2, 3, 4, and 5. If you draw this clock you will see that $4 + 3 \equiv 1 \pmod{6}$.

Exercises 12-1

1. Copy and complete the table for addition (mod 4). Use it to complete the following number sentences:

   (a) $1 + 3 \equiv ? \pmod 4$ (c) $2 + 2 \equiv ? \pmod 4$

   (b) $3 + 3 \equiv ? \pmod 4$ (d) $2 + 3 \equiv ? \pmod 4$

2. Make a table for addition (mod 3) and for addition (mod 5).

3. Use the tables in Problem 2 to find the answers to the following:

   (a) $1 + 2 \equiv ? \pmod 3$ (c) $2 + 2 \equiv ? \pmod 3$

   (b) $3 + 3 \equiv ? \pmod 5$ (d) $4 + 3 \equiv ? \pmod 5$

4. Make whatever tables you need to complete the following number sentences.

   (a) $5 + 3 \equiv ? \pmod 6$ (c) $3 + 6 \equiv ? \pmod 7$

   (b) $5 + 5 \equiv ? \pmod 6$ (d) $4 + 5 \equiv ? \pmod 7$

   Note: be sure to keep all the tables you have made. You will find use for them later in this chapter.

5. Find a replacement for x to make each of the following number sentences a true statement.

   (a) $4 + x \equiv 0 \pmod 5$ (e) $3 + x \equiv 2 \pmod 5$

   (b) $x + 1 \equiv 2 \pmod 3$ (f) $x + 4 \equiv 3 \pmod 5$

   (c) $1 + x \equiv 2 \pmod 3$ (g) $x + 2 \equiv 0 \pmod 3$

   (d) $2 + x \equiv 4 \pmod 5$ (h) $4 + x \equiv 4 \pmod 5$

6. You have a five-minute clock. How many complete revolutions would the hand make if you were using it to tell when 23 minutes had passed? Where would the hand be at the end of the 23 minute interval? (Assume that the hand started from the 0 position.)

7. Seven hours after eight o'clock is what time? What new kind of addition did you use here?

8. Nine days after the 27th of March is what date? What new kind of addition did you use here?

12-2. A New Kind of Multiplication.

Before considering a new multiplication let us look at a part of a multiplication table for the whole numbers. Here it is:

	x	0	1	2	3	4	5	↓ 6
	0	0	0	0	0	0	0	0
	1	0	1	2	3	4	5	6
	2	0	2	4	6	8	10	12
	3	0	3	6	9	12	15	18
	4	0	4	8	12	16	20	24
→	5	0	5	10	15	20	25	30
	6	0	6	12	18	24	30	36

If we had this table and forgot what 5 times 6 is equal to, we could look in the row labeled 5 and the column labeled 6 and find the answer, 30, in the 5-row and 6-column (see arrows above). Of course it is easier to memorize the table since we use it so frequently, but if we had not memorized it, it might be a very convenient thing to have in our pocket for easy reference.

How would you make such a table if you didn't know it already? This would be quite easy if you could add. The first line is very easy — you write a row of zeros. For the second line you merely have to know how to count. For the third line you add 2 each time; for the fourth line add 3 each time, and so forth.

Now, if we use the same method, we can get a multiplication table (mod 4). First block it out, filling in the first and second rows and columns:

(Mod 4)

x	0	1	2	3
0	0	0	0	0
1	0	1	2	3
→ 2	0	2		
3	0	3		

We have just four blanks to fill in. To get the 2-row (indicated by the arrow above), we add twos. Thus the third entry (which is 2 × 2) is $2 + 2 \equiv 0 \pmod 4$. Then our first three entries will look like this:

2	0	2	0

To get the fourth entry, we add 2 to the third entry. Since $0 + 2 \equiv 2 \pmod 4$, the complete 2-row is now

2	0	2	0	2.

For the last row we will have to add threes. Here $3 + 3 \equiv 2 \pmod 4$ and $3 + 2 \equiv 1 \pmod 4$. So the complete table is:

x	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

Now consider one way in which this table could be used. Suppose a lamp has a four-way switch so that it can be turned to one of four positions: off, low, medium, high. We might let numbers correspond to these positions as follows:

off	low	medium	high
0	1	2	3

If the light were at medium and we flicked the switch three times, the light would be at the low position since $2 + 3 \equiv 1 \pmod 4$. Suppose the light were off and three people flicked the switch three times each; what would be the final position of the light? The answer would be "low" since $3 \cdot 3 \equiv 1 \pmod 4$ and the number 1 corresponds to "low."

Consider an application of another multiplication table. A jug of juice lasts three days in the Willcox family. One Saturday, Mrs. Willcox bought six jugs which the family started using on the following day. What day of the week would it be necessary for her to purchase juice again? Of course it would be possible to count on one's fingers so to speak: 3 days after Saturday is Tuesday, 3 days after Tuesday is Friday, etc. But it is much easier if we notice that since there are seven days in the week, this is connected with multiplication (mod 7). We could let the number 0 correspond to Saturday since this is the day we start with and so on as follows:

Sat.	Sun.	Mon.	Tue.	Wed.	Thur.	Fri.
0	1	2	3	4	5	6

Now we need to find what $6 \cdot 3$ is (mod 7). We do not need the complete multiplication table since we are trying to find a multiple of 6. So we compute the 6-row in the usual way by adding sixes, using the addition table (mod 7) which we constructed for Problem 4 of the previous set of exercises.

(mod 7)

x	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
.	.	.	.	.	.	.	.
6	0	6	5	4	3	2	1

This means that $6 \cdot 3 \equiv 4 \pmod{7}$ and since Wednesday corresponds to 4, it follows that Mrs. Willcox would next have to buy juice on a Wednesday. Actually we did not really need to construct all the 6-row.

Exercises 12-2

1. (a) Make a table for multiplication (mod 5).
   (b) Make a table for multiplication (mod 7).
   (c) Make a table for multiplication (mod 6).
   Note: Keep these tables for future use.

2. Complete the following number sentences to make them true statements. You may find the tables you constructed in Problem 1 useful.
   (a) $3 \times 2 \equiv ? \pmod{5}$
   (b) $3 \times 4 \equiv ? \pmod{6}$
   (c) $6 \times 4 \equiv ? \pmod{7}$
   (d) $1 + (3 \times 4) \equiv ? \pmod{6}$
   (e) $5 + (6 \times 5) \equiv ? \pmod{7}$

3. Find a replacement for x to make each of the following number sentences a true statement: (Draw the clocks which you need.)
   (a) $5 \cdot 10 \equiv x \pmod{11}$
   (b) $7 \cdot 13 \equiv x \pmod{15}$
   (c) $(3 \cdot 4) + 2 \equiv x \pmod{8}$
   (d) $(4 \cdot 7) + 11 \equiv x \pmod{13}$

4. (a) Find the date of 10 weeks after December fourth.
   (b) In 1957, August sixth was a Tuesday. What day of the week was August sixth in 1959?

5. Form a table of remainders after division by 5, where the entry in any row and column is the remainder after the product is divided by 5. For instance, since the remainder is 1 when $2 \cdot 3$ is divided by 5, we will have a 1 in the 2-row and 3-column (see arrows). We have written in a few entries to show how it goes. For instance, to get the entry in the 2-row and 4-column we multiply 2 by 4 to get 8 and since the remainder when 8 is divided by 5 is 3, we put 3 in the 2-row and 4-column. Complete the table:

	0	1	2	↓ 3	4
0	0	0	0	0	0
1	0	1	2	3	4
→ 2	0	2		1	3
3	0	3		4	
4	0	4	3		

6. Do you notice any relationship between the table of Problem 5 and another you have found? Can you give any reason for this? How can this be used to make the solution of some of the problems simpler?

*7. Use the multiplication table (mod 5) to find the replacement for x to make each of the following number sentences a true statement:

(a) $3x \equiv 1 \pmod 5$

(b) $3x \equiv 2 \pmod 5$

(c) $3x \equiv 3 \pmod 5$

(d) $3x \equiv 4 \pmod 5$

(e) $3x \equiv 0 \pmod 5$

*8. If it were (mod 6) instead of (mod 5) in the previous problem, would you be able to find x in each case? If not, which equivalences would give some value of x?

12-3. What is an Operation?

We are familiar with the operations of ordinary arithmetic--addition, multiplication, subtraction and division of numbers. In Section 12-1, a different operation was discussed. We made a table for the new type of addition of the numbers 0, 1, 2, 3. This operation is completely described by the table that you made in Problem 1 of Exercises 12-1. That is, there are no numbers to which the operation is applied except those indicated and the results of the operation on all pairs of these numbers are given. The table tells what numbers can be put together. For instance, the table tells us that the number 5 cannot be combined with any number in the new type of addition since "5" does not appear in the left column nor in the top row. It also tells us that $2 + 3 \equiv 1 \pmod 4$. Study the following tables:

(a)

+	1	2	3	4	5
1	2	3	4	5	1
2	3	4	5	1	2
3	4	5	1	2	3
4	5	1	2	3	4
5	1	2	3	4	5

(b)

+	3	5	7	9
3	6	8	10	12
5	8	10	12	14
7	10	12	14	16
9	12	14	16	18

(c)

□	0	1	2	3
0	0	1	2	3
1	2	3	4	5
2	4	5	6	7
3	6	7	8	9

(d)

⊙	1	2	3
1	3	1	2
2	1	2	3
3	2	3	1

(e)

Δ	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	1	3	5
3	3	6	2	5	1	4
4	4	1	5	2	6	3
5	5	3	1	6	4	2
6	6	5	4	3	2	1

So far the only operations we have had have been called multiplication or addition. Here in (c), (d) and (e) we have different operations and so we use different symbols: □, ⊙ and Δ .

From each one of these tables we can find a certain set (the set of elements in the left column and top row) and we can put any two elements of this set together to get one and only one thing. For instance, in Table (a), the set is {1, 2, 3, 4, 5} since these are the numbers which appear in the left column and top row. These are the only numbers which can be put together by Table (a). In Table (b), the set is {3, 5, 7, 9}. What set is given by Table (c)? by Table (d)? by Table (e)?

Here are some examples from the tables:

3 + 5 = 3 in Table (a),
3 + 5 = 8 in Table (b),
2 □ 1 = 5 and 2 □ 2 = 6 in Table (c). Read ""2 square 1 equals 5".
1 ⊙ 1 = 3 in Table (d). Read "1 circle-dot 1 equals 3."
5 Δ 2 = 3 in Table (e). Read "5 triangle 2 equals 3."

In each case we had a set of elements: in (a) the set is $\{1, 2, 3, 4, 5\}$; in (d) it was $\{1, 2, 3\}$. We also had an operation: in (a) it was + ; in (d) it was $\odot$. Finally we had the result of combining any two elements by means of the operation; in (a) the results were 1, 2, 3, 4 or 5; in (c) they were 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. All of these operations are called binary operations because they are applied to two elements to get a third. So far the elements have been numbers but we shall see later that they do not need to be.

The two elements which we combine may be the same one and the result of the operation may or may not be an element of the set but it must be something definite - not one of several possible things.

You are already familiar with some operations defined on the set of whole numbers.

Any two whole numbers can be added. Addition of 8 and 2 gives 10.

Any two whole numbers can be multiplied. Multiplication of 8 and 2 gives 16.

Addition and multiplication are two different operations defined on the set of whole numbers.

In discussing subtraction, for instance with whole numbers, it is convenient to look ahead to the work of the eighth grade. The expression "6 - 9" is not the name of anything you have studied in this course. That is, it is not now possible for us to combine 6 and 9 (in that order) by subtraction and get "a definite thing" and you may wonder whether or not subtraction is an operation. Next year you will learn that there is "a definite thing" (in fact, a number) which is called "6 - 9." With this in mind, we will consider subtraction a binary operation defined on the whole numbers (or rational numbers, etc.), even though we are not yet acquainted with all the results obtained from subtraction.

When an operation is described by a table, the elements of the set are written in the same order in the top row (left to right) and in the left column (top to bottom). Keeping the order the same will make some of our later work easier.

We must also be careful about the order in which two elements are combined. For example,

$$2 \square 1 = 5, \quad \text{but} \quad 1 \square 2 = 4.$$

For this reason, we must remember that when the procedure for reading a table was explained, it was decided to write the element in the left column first and the element in the top row second with the symbol for the operation between them. We must examine each new operation to see if it is commutative and associative. These properties have been discussed in previous chapters; they are briefly reviewed here.

An operation $+$ defined on a set is called commutative if, for any elements, a, b, of the set, $a + b = b + a$.

An operation $+$ defined on a set is called associative if any elements, a, b, c, of the set can be combined as $(a + b) + c$, and also as $a + (b + c)$, and the two results are the same: $(a + b) + c = a + (b + c)$.

Exercises 12-3

1. Use the tables on pages 536 and 537 to answer the following questions:
    - (a) $3 + 3 = ?$ if we use Table (a).
    - (b) $3 + 3 = ?$ if we use Table (b).
    - (c) $3 \square 2 = ?$
    - (d) $2 \square 3 = ?$
    - (e) $2 \odot 2 = ?$
    - (f) $1 \odot 1 = ?$

(g) $(2 \odot 3) \odot 3 = ?$

(h) $2 \odot (3 \odot 3) = ?$

(i) $(1 \square 1) \square 2 = ?$

(j) $1 \square (1 \square 2) = ?$

(k) $2 \triangle (3 \triangle 4) = ?$

(l) $(2 \triangle 3) \triangle 4 = ?$

2. (a) Which of the binary operations described in the tables in this section are commutative?

   (b) Is there an easy way to tell if an operation is commutative when you examine the table for the operation? What is it?

3. How can you tell if an operation is associative by examining a table for the operation? Do you think the operations described in the tables in this section are associative?

4. Are the following binary operations commutative? Make at least a partial table for each operation. Which ones do you think are associative?

   (a) Set: All counting numbers between 25 and 75.
   Operation: Choose the smaller number.
   Example: 28 combined with 36 produces 28.

   (b) Set: All counting numbers between 500 and 536.
   Operation: Choose the larger number.
   Example: 520 combined with 509 produces 520.

   (c) Set: The prime numbers.
   Operation: Choose the larger number.

   (d) Set: All even numbers between 39 and 61.
   Operation: Choose the first number.
   Example: 52 combined with 46 produces 52.
   46 combined with 52 produces 46.

(e) Set: All counting numbers less than 50.
Operation: Multiply the first by 2 and then add the second.
Example: 3 combined with 5 produces 11.
Since $2 \cdot 3 + 5 = 11$.

(f) Set: All counting numbers.
Operation: Find the greatest common factor.
Example: 12 combined with 18 produces 6.

(g) Set: All counting numbers.
Operation: Find the least common multiple.
Example: 12 combined with 18 produces 36.

(h) Set: All counting numbers.
Operation: Raise the first number to a power whose exponent is the second number.
Example: 5 combined with 3 produces 5^3.

5. Make a table for an operation that has the commutative property.

6. Make up a table for an operation that does not have the commutative property.

We have been discussing binary operations. The word "binary" indicates that two elements are combined to produce a result. There are other kinds of operations. A result might be produced from a single element, or by combining three or more elements. When we have a set and, from any one element of the set, we can determine a definite thing, we say there is a "unary operation" defined on the set. If we combined three elements to produce a fourth we would call it a "ternary operation". One example of a ternary operation would be finding the G.C.F. of three counting numbers: e.g. the G.C.F. of 6, 8, and 10 is 2.

*7. Try to show a way of describing the following unary operation by some kind of a table?

Set: All the whole numbers from 0 to 10.

Unary Operation: Cube the number.

Example: Doing the operation to 5 produces $5^3 = 125$.

12-4. Closure.

(a)

+	1	2	3	4	5
1	2	3	4	5	1
2	3	4	5	1	2
3	4	5	1	2	3
4	5	1	2	3	4
5	1	2	3	4	5

(b)

+	3	5	7	9
3	6	8	10	12
5	8	10	12	14
7	10	12	14	16
9	12	14	16	18

(c)

□	0	1	2	3
0	0	1	2	3
1	2	3	4	5
2	4	5	6	7
3	6	7	8	9

(d)

⊙	1	2	3
1	3	1	2
2	1	2	3
3	2	3	1

Study the Tables (a) and (b). In Table (a) the results of performing the operation are the numbers which were combined by the operation (1, 2, 3, 4, 5 over again). But in Table (b) the results of performing the operation were different numbers from those combined (6, 8, 10, etc. instead of 3, 5, 7, 9). We have seen this kind of difference before, and we have a name for it. We have said that the set of whole numbers is "closed under addition" because if any two whole numbers are combined by adding them, the result is a whole number. In the same way the set {1, 2, 3, 4, 5} in Table (a) is closed under the new type of addition there since the results of the operation are again in the same set.

However, the set of odd numbers is not closed under addition since the result of adding two odd numbers is not an odd number. In the same way, in Table (b) the set 3, 5, 7, 9 is not closed under the operation of addition given there since the result is not one of the set {3, 5, 7, 9}.

Example 1:

The set of whole numbers: {1, 2, 3, 4} is not closed under multiplication because $2 \cdot 3 = 6$ which is not one of the set. Of course $1 \cdot 2 = 2$ is in the set but for a set to be closed the result must be in the set no matter what numbers of the set are combined.

Example 2:

The set of all whole numbers is closed under multiplication because the product of any two whole numbers is a whole number again.

Example 3:

The set of whole numbers is not closed under subtraction. For example, consider the two whole numbers 6 and 9. There are two different ways we can put these two numbers together using subtraction: $9 - 6$ and $6 - 9$. The first numeral, "9 - 6", is a name for the whole number 3, but the numeral "6 - 9" is not the name of any whole number. Thus, subtracting two whole numbers does not always give a whole number.

Example 4:

The set of counting numbers is not closed under division. It is true that $\frac{8}{2} = 8 \div 2$ is a counting number, but there is no counting number $\frac{9}{2}$. Can you give some other illustrations of closure, that is, sets closed under an operation and sets not closed under an operation?

Example 5:

What can we say about a set S of counting numbers which is closed under addition and which contains the number 3? What other numbers must it contain? Since 3 is in S, $3 + 3$, or 6, must also be in S. Since $(3 + 3)$ and 3 are members of S, $(3 + 3) + 3 = 6 + 3 = 9$ must be in S. Since $(3 + 3 + 3)$ and 3 are in S, $(3 + 3 + 3) + 3 = 9 + 3 = 12$ must also be in S. We can continue adding 3 to the resulting numbers to see that $3k$ must be in S for any counting number k. Thus S must contain all of the multiples of 3. What is the *smallest* set S of counting numbers containing 3 and closed under addition? As we have seen S must contain all multiples of 3. What if S contains *only* these numbers:

$$S = \{3, 6, 9, 12, \ldots\}.$$

Is S closed under addition? Is the sum of any two multiples of 3 a multiple of 3? If k and m are counting numbers, is $3k + 3m$ a multiple of 3? The answer is yes, of course, since, by the distributive property of multiplication over addition,

$$3k + 3m = 3(k + m).$$

Thus, $S = 3, 6, 9, 12, \ldots$ is closed under addition, and is the *smallest* set closed under addition, which contains 3. We call S the set *generated* by 3 under addition.

Example 6:

We could ask the same questions about multiplication which we asked about addition in Example 5. What is the smallest set closed under multiplication and containing 3?

Such a set certainly must contain

$$\begin{aligned}&3,\\&3\cdot 3 = 3^2,\\&3^2\cdot 3 = (3\cdot 3)\cdot 3 = 3^3,\\&3^3\cdot 3 = [(3\cdot 3)\cdot 3]\cdot 3 = 3^4,\end{aligned}$$

and so on.

That is, every number 3^k, where k is a counting number, must be in the set. Is the set $T = \{3, 3^2, 3^3, \ldots\}$ closed under multiplication? If n and m are counting numbers, is $3^n \cdot 3^m$ a member of T?

Write

$$3^n = \overbrace{3\cdot 3\cdot \ldots \cdot 3}^{n \text{ factors}},$$

$$3^m = \overbrace{3\cdot 3\cdot \ldots \cdot 3}^{m \text{ factors}},$$

$$3^n\cdot 3^m = \overbrace{3\cdot 3\cdot \ldots\cdot 3}^{n}\cdot\overbrace{3\cdot 3\cdot \ldots\cdot 3}^{m} = \overbrace{3\cdot 3\cdot \ldots\cdot 3}^{n+m}$$

$$= 3^{n+m}.$$

Thus $3^n \cdot 3^m = 3^{n+m}$ is a power of 3 also, so if 3^n and 3^m are in T so is their product. Thus

$$T = \{3, 3^2, 3^3, 3^4, \ldots\}$$

is the smallest set closed under multiplication and containing 3. We call T the set generated by 3 under multiplication.

Following Examples 5 and 6 we say that the set generated by an element a under an operation $*$ is the set $\{a, a * a, (a * a) * a, [(a * a) * a] * a, \ldots\}$.

Exercises 12-4

1. Study again Tables (a) - (d) in this section, page 542. Which tables determine a set that is closed under the operation? Which tables determine a set that is not closed under the operation? How do you know?

2. Which of the sets below are closed under the corresponding operations?
    - (a) The set of even numbers under addition.
    - (b) The set of even numbers under multiplication.
    - (c) The set of odd numbers under multiplication.
    - (d) The set of odd numbers under addition.
    - (e) The set of multiples of 5 under addition.
    - (f) The set of multiples of 5 under subtraction.
    - (g) The set $\{1, 2, 3, 4\}$ under multiplication (mod 5).
    - (h) The set of counting numbers less than 50 under the operation of choosing the smaller number.
    - (i) The set of prime numbers under addition.
    - (j) The set of numbers whose numerals in base five end in "3" under addition.
3. Find the smallest set of counting numbers which is;
    - (a) Closed under addition and containing 2.
    - (b) Closed under multiplication and containing 2.
4. (a) Find the set generated by 7 under addition.
    - (b) Find the set generated by 7 under multiplication.
5. Let S be the set determined by Table (d) in this section. Find the subset of S which is generated by 1 under $\odot$. Find the subset of S which is generated by 2 under $\odot$.

*6. What subset of the set of rational numbers is generated by 3? Is this set closed under division? (Is 3 in the set? Is $\frac{1}{3}$ in the set? Is $3 \div \frac{1}{3}$ in the set?) Does $(3 \div 3) \div 3 = 3 \div (3 \div 3)$? Is the division operation associative?

*7. If an operation defined on a set is commutative, must the set be closed under the operation?

*8. If an operation defined on a set is associative, must the set be closed under the operation?

*9. Make up a table for an operation defined on the set $\{0, 43, 100\}$ so that the set is closed under the operation.

*10. Make up a table for an operation defined on the set $\{0, 43$ $100\}$ so that the set is not closed under the operation.

12-5. Identity Element; Inverse of an Element.

In our study of the number one in ordinary arithmetic, we observed that the product of any number and 1 (in either order) is the same number.

For instance,

$$2 \times 1 = 2, \quad 1 \times 2 = 2, \quad 156 \times 1 = 156, \quad 1 \times 156 = 156.$$

For any number n in the arithmetic of rational numbers, $n \cdot 1 = n$ and $1 \cdot n = n$.

In our study of the number zero in the arithmetic of rational numbers we observed that the sum of 0 and any number (in either order) gave that same number; that is the sum of any number and 0 is the number. For instance,

$$2 + 0 = 2, \qquad 0 + 2 = 2, \qquad 468 + 0 = 468, \qquad 0 + 468 = 468$$

For any number n in ordinary arithmetic, $n + 0 = n$ and $0 + n = n$.

One is the identity for multiplication in ordinary arithmetic.

Zero is the identity for addition in ordinary arithmetic.

Suppose we let * stand for a binary operation. Some possibilities for * are the following:

1. If * means addition of rational numbers, 0 is an identity element because $0 * a = a = a * 0$ for any rational number, a.
2. If * means multiplication of rational numbers, 1 is an identity element because $1 * a = a = a * 1$ for any rational number.
3. If * means the greater of two counting numbers, then $1 * 2 = 2$ because 2 is greater than 1; $1 * 3 = 3$ because 3 is greater than 1; $1 * 4 = 4$ since 4 is greater than 1, etc. In fact,

   $$1 * a = a = a * 1$$

   no matter what counting number a is. So 1 is the identity for this meaning of the operation *.

We could state this formally as follows: If * stands for a binary operation on a set of elements and if there is some element, call it e, which has the property that

$$e * a = a * e = a$$

for every element a of the set, then e is called an identity element of the operation *.

As another example consider the following table for an operation which we might call # .

#	A	B	C	D
A	B	C	D	A
B	C	D	A	B
C	D	A	B	C
D	A	B	C	D

Is there an identity element for # ? Could it be A? Is $A \# B = B$? (Read "A sharp B equals B"). Since, from the table $A \# B = C$, the answer to the question is "no" and we see that A cannot be the identity. Neither can B be the identity since $A \# B$ is not A. However, D is an identity for # , since

$$\begin{aligned} A \# D = D \# A &= A, \\ B \# D = D \# B &= B, \\ C \# D = D \# C &= C, \\ D \# D &= D. \end{aligned}$$

Compare the column under D with the column under the # . Compare the row to the right of D with the row to the right of the # . What do you notice? Does this suggest a way to look for an identity element when you are given a table for the operation?

If we have an identity element then we may also have what is called an inverse element. If the operation is multiplication for rational numbers, the identity is 1 and we call two rational numbers a and b inverses of each other if their product is 1, that is, if each is the reciprocal of the other.

Suppose the operation is addition (mod 4). Here 0 is the identity element and we call two numbers inverses if their sum is 0, that is, if combining the two numbers by the operation gives 0. To find inverses (mod 4) for addition, from the table:

0 is the identity
$2 + 2 \equiv 0 \pmod 4$
$3 + 1 \equiv 0 \pmod 4$
$1 + 3 \equiv 0 \pmod 4$

(Mod 4)

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

Here 0 is its own inverse, 2 is its own inverse, and 3 and 1 are inverses of each other.

Definition. Two elements a and b are inverses (or either one is the inverse of the other) under a binary operation $*$ with identity element e if $a * b = e$ and $b * a = e$.

Write again the table for # which we had in the beginning of this section.

#	A	B	C	D
A	B	C	D	A
B	C	D	A	B
C	D	A	B	C
D	A	B	C	D

Remember that we showed that D is the identity element for this table.

Can you find an element of the set {A, B, C, D} which will make the statement A # ___ = D true? It is C : A # C = D. A and C are inverses of each other under # . Can you find any other elements with inverses under # ?

Exercises 12-5a

1. Study tables (a) - (d) in Section 12-3.
   (a) Which tables describe operations having an identity and what is the identity?
   (b) Pick out pairs of elements which are inverses of each other under these operations. Does each member of the set have an inverse?
2. For each of the operations of Problem 4, Exercises 12-3;
   (a) Does the operation have an identity and, if so, what is it?
   (b) Pick out pairs of elements which are inverses of each other under these operations.

(c) For which operations does each element have an inverse?

*3. Can there be more than one identity element for a given binary operation?

If the operation is multiplication we call inverses multiplicative inverses. (The multiplicative inverse of a rational number is its reciprocal.) Consider the set of counting numbers with multiplication as the operation. What elements have multiplicative inverses? Does 5 have a multiplicative inverse in the set of counting numbers? Is $\frac{1}{5}$ a counting number? The element 5 has no multiplicative inverse in this set. Does 1 have a multiplicative inverse in this set? Yes, it does, for $1 \cdot 1 = 1$. It is the only element of this set which has a multiplicative inverse, and it is its own multiplicative inverse. Of course, if we expand the set under consideration to include all of the rational numbers except zero then each element has a multiplicative inverse. The numbers in the pairs 5 and $\frac{1}{5}$, 1 and 1, $\frac{4}{9}$ and $\frac{9}{4}$ are multiplicative inverses of each other. Does 0 have a multiplicative inverse? Is there any number b such that $0 \cdot b = 1$?

Recall the (mod 5) multiplication.

×	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

How would we decide what elements of the set $\{0, 1, 2, 3, 4\}$ have (multiplicative) inverses in this mathematical system? The identity for multiplication (mod 5) is 1. We would be looking for products which are the identity, so we should look for ones in the table. There are 4 ones in the table. They tell us that $1 \cdot 1 \equiv 1 \pmod{5}$, $2 \cdot 3 \equiv 1 \pmod{5}$, $3 \cdot \underline{\quad} \equiv 1 \pmod{5}$, and $4 \cdot \underline{\quad} \equiv 1 \pmod{5}$. (You supply the missing numbers.)

Thus the multiplicative inverse of 2 in (mod 5) is 3. What is the multiplicative inverse of 3 in (mod 5)? of 4?

Do you see any connection between multiplicative inverses and the property of closure under division? Suppose you are given a set S of numbers which is closed under multiplication and suppose a is an element of S. How would you know whether it is possible to "divide by a in S"? That is, when is it possible to divide any element of S (including a) by a and obtain another element of S?

First of all, S must contain 1, since $a \div a = 1$. For instance if S were a set of rational numbers closed under multiplication, and if $\frac{1}{2}$ were in the set, then $\frac{1}{2} \div \frac{1}{2} = 1$ would also have to be in the set.

Second, since 1 is in S, S must contain $1 \div a = \frac{1}{a}$, no matter what element of S $\underline{a}$ stands for. This means that if 2 is in S, then $\frac{1}{2}$ must also be in S. If $\frac{1}{2}$ is in S, then 2 must be in S. S cannot contain zero since $1 \div 0$ has no meaning.

Third, if b is any element of S and $\frac{1}{a}$ is in S, then

$$b \cdot \frac{1}{a} = b \div a = \frac{b}{a}$$

is an element of S. For instance, if 2 is in S and $\frac{1}{3}$ is in S, then $2 \cdot \frac{1}{3} = \frac{2}{3}$ is also in S.

If S is to be closed under division it must be possible to divide any element of S by any element of S. Thus S must contain 1, every element of S must have a multiplicative inverse in S, and we must be able to divide every element by every other element.

If the system were not commutative, $b \cdot \frac{1}{a}$ might not be equal to $\frac{1}{a} \cdot b$ which means that $\frac{b}{a}$ might mean two different things. So in this chapter we consider division only when multiplication is commutative.

We can summarize what we have learned: Let S be a set of numbers closed under multiplication where multiplication is commutative. Then:

> If S is closed under division, S contains the number 1, and every element of S has a multiplicative inverse in S. (If a is in S then $\frac{1}{a}$ is also in S.)

Also, the other way around,

> If S contains 1 and if every element of S has its multiplicative inverse in S, then S is closed under division.

Perhaps you can now see another reason why we call division the inverse operation for multiplication:

> Dividing by a number a is the same as multiplying by the multiplicative inverse of a.

For instance, if S is the set of rational numbers with zero excluded, $\frac{1}{2}$ is the multiplicative inverse (reciprocal) of 2 and hence multiplying by $\frac{1}{2}$ always gives the same result as dividing by 2. If S were the set of numbers 0, 1, 2, 3, 4 and multiplication were (mod 5) as in the table above, then, since 3 is the inverse of 2, multiplying by 3 gives the same result as dividing by 2, that is, the values of x in the two following equivalences are the same:

$$1 \equiv 2x \pmod 5 \ ; \ 1 \cdot 3 \equiv x \pmod 5.$$

In the first case $x \equiv 1 \div 2 \pmod 5$ and in the second $x \equiv 1 \cdot 3 \pmod 5$.

Everything we have said about division and multiplicative inverses can be said about subtraction and additive inverses. Here again we consider subtraction only when addition is commutative in the system. Consider first (mod 4) subtraction. What is 3 - 1 (mod 4)? Since subtraction is the inverse operation for addition, to find 3 - 1 (mod 4) we must find the missing number in the sentence ? + 1 ≡ 3 (mod 4).

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

To find the answer from the Table notice that, since the table gives sums, the 3 will be inside the table, and since 1 is the number which is added it will appear at the top of the table. If we look down the 1-column until we find a 3, we see that it is in the 2-row. So 2 is the number which, when you add 1 to it, you get 3. Since the system is commutative, the answer to 1 + ? ≡ 3 (mod 4) is also 2; that is, if we look along the 1-row until we see a 3, it will be in the 2-column. If the system were not commutative 3 - 1 would have two meanings, which would be awkward. What is 2 - 3 (mod 4)? What number must we add to 3 in (mod 4) to obtain 2? We see from the table that 3 + 3 ≡ 2 (mod 4), so 2 - 3 ≡ 3 (mod 4). What is 1 - 3 (mod 4)?

Now let us ask another kind of question. Is there any number which we can add to 2 to obtain 2 - 3 (mod 4)? Now 2 - 3 ≡ 3 (mod 4) since 2 ≡ 3 + 3 (mod 4). If we look at the table we see that 2 + 1 ≡ 3 (mod 4) and hence

$$2 - 3 \equiv 2 + 1 \pmod{4}.$$

This means that if we subtract 3 from 2 we have the same result as if we add 1 to 2. In other words, adding 1 to 2 gives the same result as subtracting its inverse, 3, from 2.

In the same way you should show that

$$1 - 3 \equiv 1 + 1 \pmod{4}$$
$$3 - 1 \equiv 3 + 3 \pmod{4}.$$

From the first two of these examples it appears that, in (mod 4), subtracting 3 produces the same result as adding 1. Is this always true in this system? Is $0 - 3 \equiv 0 + 1 \pmod{4}$? Is $3 - 3 \equiv 3 + 1 \pmod{4}$?

What is the relationship between 1 and 3 in (mod 4)? Since $1 + 3 \equiv 0 \pmod{4}$, and 0 is the identity under addition in (mod 4), what do we say about 1 and 3? They are additive inverses of each other.

Perhaps you can guess a general principle from this example. We observe that:

> Subtracting a number produces the same result as adding the additive inverse of the number.

This principle will be true in any commutative system where we call an operation "addition" and where the elements have inverses. Also, similar to a property which we have observed for multiplication, we have:

> A set which is closed under addition (where addition is commutative) will be closed under subtraction if it contains 0 and contains the additive inverse of each of its members.

Notice that in addition (mod n) we have our first examples of sets which are closed under subtraction. Nowhere in our study of the counting numbers, the whole numbers, and the rational numbers this year have we had additive inverses, except that in all these systems the number zero is its own additive inverse.

In the following exercises you will be given the chance to test these general principles further.

Exercises 12-5b

1. (a) Use the multiplication table for (mod 6) to find, wherever possible, a replacement for x to make each of the following number sentences a true statement:

$1 \cdot x \equiv 1 \pmod 6$ $\qquad 4x \equiv 1 \pmod 6$

$2x \equiv 1 \pmod 6$ $\qquad 5x \equiv 1 \pmod 6$

$3x \equiv 1 \pmod 6$

(b) Which elements of the set $\{0, 1, 2, 3, 4, 5\}$ have multiplicative inverses in (mod 6)?

2. Remember that division is defined as the inverse operation for multiplication. Thus, in the arithmetic of rational numbers, the question "Six divided by two is what?" means, really, "Six is obtained by multiplying two by what?" We can define division (mod n) in this way:

$$6 \div 4 \equiv \,?\, \pmod 5 \qquad \text{means}$$
$$(4)(?) \equiv 6 \pmod 5.$$

Copy and complete the following table using multiplication and division (mod 5).

(Mod 5)

b	a	multiplicative inverse of a	$b \div a$	$b \cdot \left(\begin{array}{c}\text{multiplicative} \\ \text{inverse of } a\end{array}\right)$
1	2	3	$1 \div 2 \equiv 3$	$1 \cdot 3 \equiv 3$
2	2	3	$2 \div 2 \equiv 1$	$2 \cdot 3 \equiv 1$
3	2	3	$3 \div 2 \equiv$	$3 \cdot 3 \equiv$
2	3			
3	3			
4	3			
1	4			
2	4			
3	4			
4	4			

3. Here is a table for addition (mod 5).

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

Copy and complete the following table.

(Mod 5)

b	a	additive inverse of a	b - a	b + additive inverse of a
0	1	4	$0 - 1 \equiv 4$	$0 + 4 \equiv 4$
2	1		$2 - 1 \equiv$	$2 + 4 \equiv$
4	1			
1	2			
2	2			
3	2			
2	4			
3	4			
4	4			

4. In the arithmetic of rational numbers which of the following sets is closed under division?

(a) $\{1, 2, \frac{1}{2}\}$

(b) $\{1, 2, 2^2, 2^3, \ldots\}$

(c) The non-zero counting numbers.

(d) The rational numbers.

5. (a) Which of the following sets is closed under multiplication (mod 6)?

{0, 1, 2, 3, 4, 5}, {2, 4}, {0, 1, 5}, {1, 5}, {5}?

(b) Which of the sets in (a) contain a multiplicative inverse (mod 6) for each of its elements?

(c) Which of the sets in (a) is closed under division (mod 6)?

6. (a) Which of the sets {A, B}, {C, D}, {B, C, D}, {A, D} is closed under the operation * defined by the table below?

*	A	B	C	D
A	A	A	A	A
B	A	A	B	B
C	A	B	D	C
D	A	B	C	D

For instance, {A, D} is closed under the operation because if we pick out that part of the table we have the little table

*	A	D
A	A	A
D	A	D

which contains only A's and D's. On the other hand the set {A, C} is not closed since its little table would be

*	A	C
A	A	A
C	A	D

Here the table contains a D, which is not one of the set {A, C}.

(b) Is there an identity for $*$? If so, what is it?

(c) Which of the sets in (a) has an inverse under $*$ for each of its elements?

*(d) Which of the sets in (a) is closed under the inverse operation for $*$? (You might use the symbols $\overset{*}{\underset{*}{-}}$ for this operation, so that $a \overset{*}{\underset{*}{-}} b = ?$ means $b * ? = a$.)

12-6. What Is a Mathematical System?

The idea of a set has been a very convenient one in this book -- some use has been made of it in almost every chapter. But there is really not a great deal that can be done with just a set of elements. It is much more interesting if something can be done with the elements (for instance, if the elements are numbers, they can be added or multiplied). If we have a set and an operation defined on the set, it is interesting to find out how the operation behaves. Is it commutative? associative? Is there an identity element? Does each element have an inverse? The "behavior" of the arithmetic operations (addition, subtraction, multiplication and division) on numbers was discussed in Chapters 3 and 6. We have seen that different operations may "behave alike" in some ways (both commutative, for instance). This suggests that we study sets with operations defined on them to see what different possibilities there are. It is too hard for us to list all the possibilities, but some examples will be given in this section and the next. These are examples of mathematical systems.

Definition. A mathematical system is a set of elements together with one or more binary operations defined on the set.

The elements do not have to be numbers. They may be any objects whatever. Some of the examples below are concerned with letters or geometric figures instead of numbers.

Example 1: Let's look at egg-timer arithmetic -- arithmetic mod 3.

(a) There is a set of elements the set of numbers {0, 1, 2}.

(b) There is an operation + mod 3, defined on the set {0, 1, 2}.

Mod 3

+	1	2	0
1	2	0	1
2	0	1	2
0	1	2	0

Therefore, egg-timer arithmetic is a mathematical system. Does this system have any interesting properties?

(c) The operation, + mod 3, has the commutative property. Can you tell by the table? If so, how? We can check some special cases too. $1 + 2 \equiv 0$ (mod 3) and $2 + 1 \equiv 0$ (mod 3), so $1 + 2 \equiv 2 + 1$ (mod 3).

(d) There is an identity for the operation + mod 3 (the number 0).

(e) Each element of the set has an inverse for the operation + mod 3.

Study the following tables.

(a)

O	A	B
A	A	B
B	A	B

(b)

*	P	Q	R	S
P	R	S	P	Q
Q	S	R	Q	P
R	P	Q	R	S
S	Q	P	S	R

(c)

~	Δ	□	○	\
Δ	Δ	□	○	\
□	□	○	\	Δ
○	○	\	Δ	□
\	\	Δ	□	○

Exercises 12-6

1. Which one, or ones, of the Tables (a), (b), (c) describes a mathematical system? Show that your answer is correct.

2. Use the tables above to complete the following statements correctly.

   (a) B O A = ?
   (b) Δ ~ ○ = ?
   (c) \ ~ \ = ?
   (d) A O B = ?
   (e) Q * R = ?
   (f) R * S = ?
   (g) P * R = ?
   (h) □ ~ ○ = ?
   (i) \ ~ □ = ?
   (j) B O B = ?
   (k) A O A = ?
   ( ) S * S = ?

3. Which one, or ones, of the binary operations O, *, ~ is commutative? Show that your answer is correct.

4. Which one, or ones, of the binary operations o, *, ~ has an identity element? What is it in each case?

5. Use the tables above to complete the following statements correctly.

(a) $P * (Q * R) = ?$

(b) $(P * Q) * R = ?$

(c) $P * (Q * S) = ?$

(d) $(P * Q) * S = ?$

(e) $(R * P) * S = ?$

(f) $R * (P * S) = ?$

(g) $\triangle \sim (\triangle \sim \backslash) = ?$

(h) $(\triangle \sim \triangle) \sim \backslash = ?$

(i) $(\bigcirc \sim \square) \sim \triangle = ?$

(j) $\bigcirc \sim (\square \sim \triangle) = ?$

6. Does either of the operations described by Table (b) or Table (c) seem to be associative? Why? How could you prove your statement? What would another person have to do to prove you wrong?

7. (a) In Table (c) what set is generated by the element $\square$?

(b) In Table (b) what set is generated by the element P?

8. BRAINBUSTER. For each of the following tables, tell why it does not describe a mathematical system.

(a)

*	1	2
1	1	1
2	1	

(b)

*	1	2
1	the product of 3 and 6	the sum of 2 and 4
2	a number between 3 and 8	0

(c)

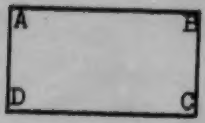
*	1	2
3	1	4
4	2	3

12-7. Mathematical Systems without Numbers.

In the last section there were some examples of mathematical systems without numbers in them. Suppose we want to invent one. What do we need?

We must have a set of things. Then, we need some kind of a binary operation -- something that can be done with any two elements of our set. We have found that the properties of closure, commutativity, associativity, etc. are very helpful in simplifying expressions. It would be nice to have some of these properties.

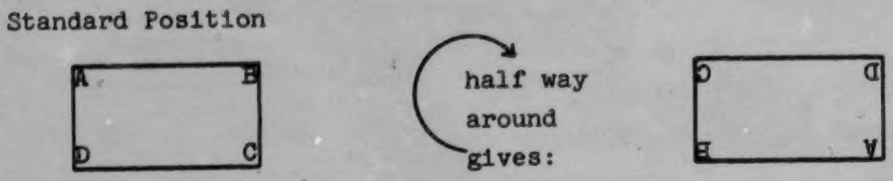
Let's start with a card. Any rectangular shaped card will do. We will use it to represent a closed rectangular region. Lay the card on your desk and label the corners as in the sketch. Now pick the card up and write the letter "A" on the other side (the side that was touching the desk) behind the "A" you have already written. Be sure the two letters "A" are back-to-back so they are labels for the same corner of the card. Similarly, label the corners B, C, and D on the other side of the card (be sure they're back-to-back with the B, C, and D you have already written.)



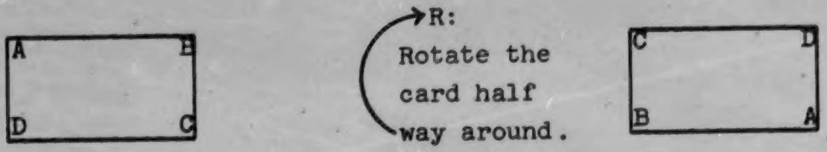
What set shall we take? Instead of numbers, let us take elements which have something to do with the card. Start with the card in the center of your desk and with the long sides of the card parallel to the front of your desk. Now move the card -- pick it up, turn it over or around in any way -- and put it back in the center of your desk with the long sides parallel to the front of your desk. The card looks just the same as it did before, but the corners may be labeled differently (a corner that started at the top may now be at the bottom, for instance). The position of the card has been changed, but the closed rectangular region looks as it did in the beginning. (The "picture" stays the same. Individual points may be moved.) The elements of our set will be these changes of position. We will take all the changes of position that make the closed rectangular region look as it did in the beginning. (Long sides parallel to the front of the desk.) How many of these changes are there?

We may start with the card in some position which we will call the standard position. Suppose it looks like the figure below.

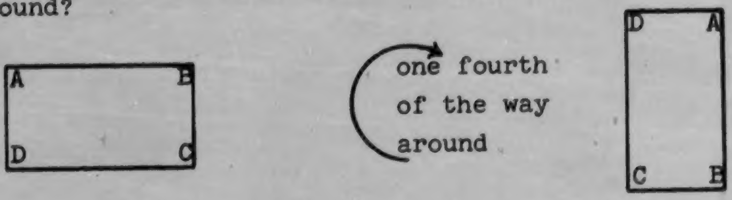
Leaving the card on your desk, rotate it half way around its center. A diagram of this change is:


Standard Position
A
B
D
C
half way
around
gives:
C
D
B
A

Since the letters "A", "B", etc. are only used as a convenience to label the different corners of the card, we will not bother to write them upside down. The diagram below represents this change of position, and we will call the change "R" (for rotation).

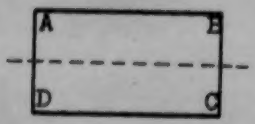

A
B
D
C
R:
Rotate the
card half
way around.
C
D
B
A

What would happen if the card were rotated one fourth of the way around?

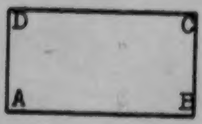

A
B
D
C
one fourth
of the way
around
D
A
C
B

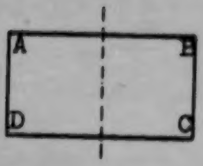
Does the card look the same before and after the change? No, this change of position cannot be in our set, since the two pictures are quite different.

Are there other changes of position of the closed rectangular region which make it look the way it did in the beginning? Yes, we can flip the card over in two different ways as shown by the diagrams below:

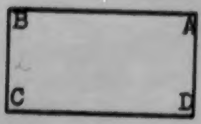


H:
Flip the card
over, using a
horizontal axis.



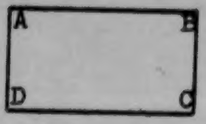


V:
Flip the card
over, using a
vertical axis.

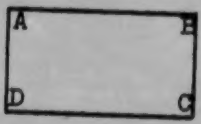


Now you know why you had to label both sides of the card so carefully. Remember, the card only represents a geometric figure for us. Turning over a card makes it different -- you see the other side; but turning over the closed rectangular region would not make it different (of course, some of the individual points would be in different positions, but the whole geometric figure would look just the same).

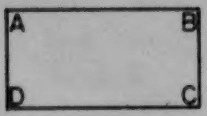
There is one more change of position which we must consider. It is the change which leaves the card alone (or puts each individual point back in place). Let us call it "I."



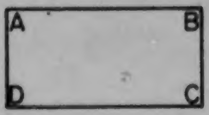
I:
Leave the card
in place.

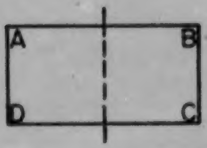


Now we have our set of elements; it is I, B, H, R. Let us summarize what they are for easy reference:

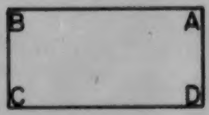


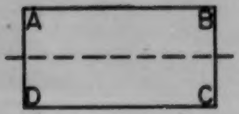
Element I:
Leave the card in place



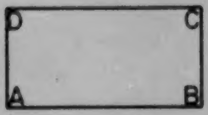


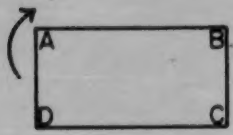
Element V:
Flip the card over using a vertical axis



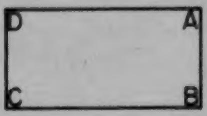


Element H:
Flip the card over using a horizontal axis





Element R:
Rotate the card halfway around in the direction indicated



Recall the definition of a mathematical system. There were two requirements:

(a) A set of elements.

(b) One or more binary operations defined on the set of elements.

Our set {I, V, H, R} satisfies the first condition. Now we need to satisfy the second condition; we need an operation. What operation shall we use? How can we "combine any two elements of our set" to get a "definite thing"? If the set is to be closed under the operation, the "definite thing" which is the result of the operation should be one of the elements again.

Here is a way of combining any two elements of our set. We will do one of the changes AND THEN do the other one. We will use the symbol "ANTH" for this operation (perhaps you can think of a better one). Thus "H ANTH V" means flip the card over, using a horizontal axis, and then flip the card over, using a vertical axis." Start with the card in the standard position and do these changes to it. What is the final position of the card? Is the result of these two changes the same as the change R? What does "V ANTH H" mean? Try it with your card. Now we can fill in the table for our operation. Some of the entries are given in the table at the right.

ANTH	I	V	H	R
I	I	V		
V			R	H
H	H	R		
R			V	

Exercises 12-7

1. Check the entries that are given in the table above and find the others. Use your card.

2. From your table for the operation ANTH, or by actually moving a card, fill in each of the blanks to make the equations correct.

   (a) R ANTH H = ?

   (b) R ANTH ? = H

   (c) ? ANTH R = H

   (d) ? ANTH H = R

(e) (R ANTH H) ANTH V = ?

(f) R ANTH (H ANTH V) = ?

(g) (R ANTH H) ANTH ? = V

(h) (R ANTH ?) ANTH V = H

(i) (? ANTH H) ANTH V = R

3. Examine the table for the operation ANTH.

(a) Is the set closed under the operation?

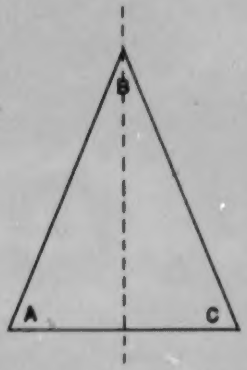
(b) Is the operation commutative?

(c) Do you think the operation is associative? Use the operation table to check several examples.

(d) Is there an identity element for the operation ANTH?

(e) Does each element of the set have an inverse under the operation ANTH?

4. Here is another system of changes. Cut a triangular card with two equal sides. Label the corners as in the sketch (both sides, back-to-back). The set for the system will consist of two changes. The first change, called I, will be: Leave the card in place. The second change, called F, will be: Flip the card over, using the vertical axis. F ANTH I will mean: Flip the card over, using the vertical axis, and then leave the card in place. How will the card look -- as if it had been left in place, I, or as if the change F had been done? What does I ANTH F mean? Does R ANTH I = F or does F ANTH I = I?



(a) Complete the table below:

ANTH	I	F
I		
F		

(b) Is the set closed under this operation?

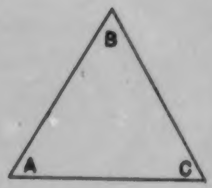
(c) Is the operation commutative?

(d) Is the operation associative? Are you sure?

(e) Is there an identity for the operation?

(f) Does each element of the set have an inverse under the operation?

5. Make a triangular card with three equal sides and label the corners as in the sketch (both sides, back-to-back). The set for this system will be made up of these six changes.



I: Leave the card in place.

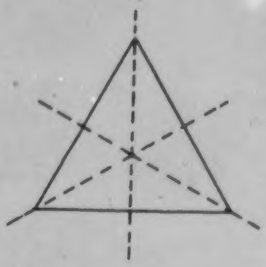
R: Rotate the card clockwise $\frac{1}{3}$ of the way around.

S: Rotate the card clockwise $\frac{2}{3}$ of the way around.

T: Flip the card over, using a vertical axis.

U: Flip the card over, using an axis through the lower right vertex.

V: Flip the card over, using an axis through the lower left vertex. Three of these will be rotations about the center (leave in place and two others). The other three will be flips about the axes. (Caution: the



axes are stationary; they do not rotate with the card. For example, the vertical axis remains vertical -- it would go through a different corner of the card after rotating the card one third of the way around its center.) Make a table for these changes. Examine the table. Is this operation commutative? Is there an identity change? Does each change have an inverse?

*6. Try making a table of changes for a square card. There are eight changes (that is, eight elements). What are they? Is there an identity element? Is the operation ANTH commutative?

12-8. The Counting Numbers and the Whole Numbers.

The mathematical systems that we have studied so far in this chapter are composed of a set and one operation. Examples are modular addition or multiplication and the changes of a rectangular or triangular card. A mathematical system given by a set and two operations would appear to be more complicated than these examples. However, as you may have guessed, ordinary arithmetic is also a mathematical system and we know that we can do more than one operation using the same set of numbers -- for examples, we can add and multiply.

To be definite, let us choose the set of rational numbers. This set, together with the two operations of addition and multiplication forms a mathematical system which was discussed in Chapters 6 and 8. Are there properties of this system which are entirely different from those we have considered in systems with only one operation? Yes, you are familiar with the fact that $2 \cdot (3 + 5) = (2 \cdot 3) + (2 \cdot 5)$. This is an illustration of the distributive property. More precisely, it illustrates that multiplication distributes over addition. The distributive property is also of interest in other mathematical systems.

Definition. Suppose we have a set and two binary operations, * and o, defined on the set. The operation * distributes over the operation o if a * (b o c) = (a * b) o (a * c) for any elements a, b, c, of the set. (And we can perform all these operations.)

In a mathematical system with two operations, there are the properties which we previously discussed for each of these operations separately. The only property which is concerned with both operations together is the distributive property.

Exercises 12-8

1. Consider the set of counting numbers:

   (a) Is the set closed under addition? under multiplication? Explain.

   (b) Do the commutative and associative properties hold for addition? for multiplication? Give an example of each.

   (c) What is the identity element for addition? for multiplication?

   (d) Is the set of counting numbers closed under subtraction? under division? Explain.

   The answers to (a), (b), and (c) tell us some of the properties of the mathematical system composed of the set of counting numbers and the operations of addition and multiplication.

2. Answer the questions of Problem 1 (a), (b) (c) for the set of whole numbers. Are your answers the same as for the counting numbers?

3. (a) For the system of whole numbers, write three number sentences illustrating that multiplication distributes over addition.

   (b) Does addition distribute over multiplication? Try some examples.

4. The two tables below describe a mathematical system composed of the set {A, B, C, D} and the two operations * and o.

*	A	B	C	D
A	A	A	A	A
B	A	B	A	B
C	A	A	C	C
D	A	B	C	D

o	A	B	C	D
A	A	B	C	D
B	B	B	D	D
C	C	D	C	D
D	D	D	D	D

(a) Do you think * distributes over o? Try several examples.

(b) Do you think o distributes over *? Try several examples.

5. Answer these questions for each of the following systems. Is the set closed under the operation? Is the operation commutative? associative? Is there an identity? What elements have inverses?

(a) The system whose set is the set of odd numbers and whose operation is multiplication.

(b) The system whose set is made up of zero and the multiples of 3, and whose operation is multiplication.

(c) The system whose set is made up of zero and the multiples of 3 and whose operation is addition.

(d) The system whose set is made up of the rational numbers between 0 and 1 (not including 0 and 1), and whose operation is multiplication.

(e) The system whose set is made up of the even numbers and whose operation is addition. (Zero is an even number.)

(f) The system whose set is made up of the rational numbers between 0 and 1, and whose operation is addition.

6. (a) In what ways are the systems of 5(b) and 5(c) the same
   (b) In what ways are the systems of 5(a) and 5(b) different?

*7. Make up a mathematical system of your own that is composed of a set and two operations defined on the set. Make at least partial tables for the operations in your system. List the properties of your system.

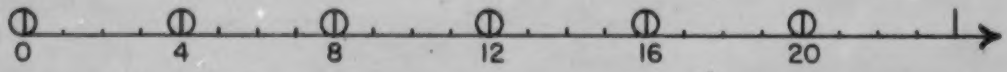
*8. Here is a mathematical system composed of a set and two operations defined on that set.
Set: All counting numbers.
Operation *: Find the greatest common factor.
Operation o: Find the least common multiple.
   (a) Does the operation * seem to distribute over the operation o? Try several examples.
   (b) Does the operation o seem to distribute over the operation *? Try several examples.

12-9. Modular Arithmetic.

In Section 12-1 we studied a new addition done by rotating the hand of a clock. Using a four-minute clock, we said that $2 + 3 = 1 \pmod 4$. The tables which we made described the mathematical system (mod 4). In Section 12-2 we studied a new multiplication using the same clock.

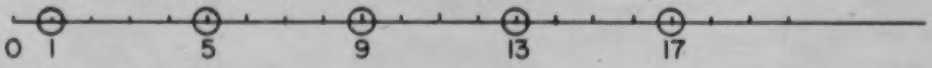
Modular systems are the result of classifying whole numbers in a certain way. For example, we could classify whole numbers as even or odd. In this case, the even numbers: 0, 2, 4, 6, ... are put in the same family and the family is named by its smallest member: 0. Thus the class of all even numbers is 0 (mod 2). Starting from 1, the odd numbers: 1, 3, 5, 7, ... are put in the same family which we call 1 (mod 2). For the odds and evens, we then have two classes, 0 (mod 2) and 1 (mod 2). The number 5 belongs to the class 1 (mod 2), 8 belongs to the class 0 (mod 2).

If we put every fourth whole number in the same class, we have the (mod 4) system. Here is a sketch of some of the numbers belonging to the class 0 (mod 4).



Every fourth whole number starting with 0 belongs to the same class. Thus, numbers which are multiples of 4 belong to the class 0 (mod 4).

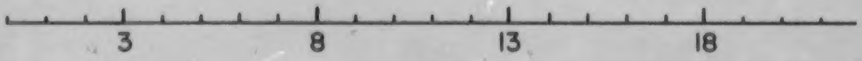
Here is a sketch showing some of the numbers which belong to the class 1 (mod 4).



Every fourth whole number starting with 1 belongs to the same class, that is, 1 (mod 4). Thus the numbers which are 1 plus a multiple of 4 belong to this class.

The two sketches below show respectively some of the numbers which belong to the class 0 (mod 5) and the class 3 (mod 5).

The numbers belonging to the class 0 (mod 5) are multiples of 5.



The numbers belonging to the class 3 (mod 5) are 3 plus multiples of 5.

Our first problems in a modular system used the operation of addition. When we changed the operation to multiplication, we got a different mathematical system. With both operations, modular arithmetic is more like ordinary arithmetic than it was with just one operation.

For each of the modular systems we can state the number of elements in the set. For instance, there are four elements if it is (mod 4), seven elements if it is (mod 7), and so forth. Such a set is called a finite set and the system is called a finite system. The modular systems and the systems of Section 12-7 are finite systems. On the other hand, the set of rational numbers considered in Section 12-8 is so large that it contains more elements than any number you could name. Such a set is called an infinite set and the system is called an infinite system.

Exercises 12-9

1. Write the multiplication table (mod 8) and recall or write again the multiplication table (mod 5) which you found in Exercises 12-2.

2. Answer each of the following questions about the mathematical systems of multiplication (mod 5) and (mod 8).

   (a) Is the set closed under the operation?

   (b) Is the operation commutative?

   (c) Do you think the operation is associative?

   (d) What is the identity element?

   (e) Which elements have inverses, and what are the pairs of inverse elements?

   (f) Is it true that if a product is zero at least one of the factors is zero?

3. Complete each of the following number sentences to make it a true statement.

   (a) $2 \times 4 \equiv ? \pmod{5}$

   (b) $4 \times 3 \equiv ? \pmod{5}$

   (c) $5^2 \equiv 1 \pmod{?}$

   (d) $2^3 \equiv 0 \pmod{?}$

4. Find the products:

(a) $2 \times 3 \equiv ?\ (\text{mod } 4)$

(b) $2 \times 3 \equiv ?\ (\text{mod } 6)$

(c) $5 \times 8 \equiv ?\ (\text{mod } 7)$

(d) $3 \times 4 \times 6 = ?\ (\text{mod } 9)$

(e) $4^3 \equiv ?\ (\text{mod } 5)$

(f) $6^2 \equiv ?\ (\text{mod } 5)$

*(g) $6^{256} \equiv ?\ (\text{mod } 5)$

5. Find the sums:

(a) $1 + 3 \equiv ?\ (\text{mod } 5)$

(b) $4 + 3 \equiv ?\ (\text{mod } 5)$

(c) $2 + 4 \equiv ?\ (\text{mod } 5)$

(d) $4 + 4 \equiv ?\ (\text{mod } 5)$

6. (a) Find the values of $3(2 + 1)\ (\text{mod } 5)$ and $(3 \cdot 2) + (3 \cdot 1)\ (\text{mod } 5)$.

(b) Find the values of $4(3 + 1)\ (\text{mod } 5)$ and $(4 \cdot 3) + (4 \cdot 1)\ (\text{mod } 5)$.

(c) Find the values of $(3 \cdot 2) + (3 \cdot 4)\ (\text{mod } 5)$ and $3(2 + 4)\ (\text{mod } 5)$.

(d) In the examples of this problem is multiplication distributive over addition?

7. (a) Find the values of $3 + (2 \cdot 1)\ (\text{mod } 5)$ and $(3 + 2) \cdot (3 + 1)\ (\text{mod } 5)$.

(b) Find the values of $4 + (3 \cdot 1)\ (\text{mod } 5)$ and $(4 + 3) \cdot (4 + 1)\ (\text{mod } 5)$.

(c) Find the values of $(3 + 2) \cdot (3 + 4)\ (\text{mod } 5)$ and $3 + (2 \cdot 4)\ (\text{mod } 5)$.

(d) In the examples of this problem is addition distributive over multiplication?

Remember that division is defined after we know about multiplication. Thus, in ordinary arithmetic, the question "Six divided by 2 is what?" means, really "Six is obtained by multiplying 2 by what?" An operation that begins with one of the numbers and the "answer" to another binary operation and asks for the other number, is called an inverse operation. Division is the inverse of the multiplication operation.

*8. Find the quotients?

(a) $2 \div 3 \equiv ? \pmod 8$

(b) $6 \div 2 \equiv ? \pmod 8$

(c) $0 \div 2 \equiv ? \pmod 8$

(d) $3 \div 4 \equiv ? \pmod 5$

(e) $0 \div 2 \equiv ? \pmod 5$

(f) $0 \div 4 \equiv ? \pmod 5$

(g) $7 \div 3 \equiv ? \pmod{10}$

*(h) $7 \div 6 \equiv ? \pmod 8$

9. Find the following; remember that subtraction is the inverse operation of addition.

(a) $7 - 3 \pmod 8$

(b) $3 - 4 \pmod 5$

(c) $3 - 4 \pmod 8$

*(d) $4 - 9 \pmod{12}$

10. Make a table for subtraction mod 5. Is the set closed under the operation?

11. Find a replacement for x which will make each of the following number sentences a true statement. Explain.

(a) $2x \equiv 1 \pmod 5$

(b) $3x \equiv 1 \pmod 4$

(c) $3x \equiv 0 \pmod 5$

(d) $3x \equiv 0 \pmod 6$

(e) $x \cdot x \equiv 1 \pmod 8$

(f) $4x \equiv 4 \pmod 8$

12. In Problem 11 (d) and (f), find at least one other replacement for x which makes the number sentence a true statement.

12-10. Summary and Review.

A binary operation defined on a set is a rule of combination by means of which any two elements of the set may be combined to determine one definite thing.

A mathematical system is a set together with one or more binary operations defined on that set.

A set is closed under a binary operation if every two elements of the set can be combined by the operation and the result is always an element of the set.

An identity element for a binary operation defined on a set is an element of the set which does not change any element with which it is combined.

Two elements are inverses of each other under a certain binary operation if the result of this operation on the two elements is an identity element for that operation.

A binary operation is commutative if, for any two elements, the same result is obtained by combining them first in one order, and then in the other.

A binary operation is $*$ associative if, for any three elements, the result of combining the first with the combination of the second and third is the same as the result of combining the combination of the first and second with the third.

$$a * (b * c) = (a * b) * c.$$

The binary operation $*$ distributes over the binary operation o provided

$$a * (b \circ c) = (a * b) \circ (a * c)$$

for all elements a, b, c.

A set S is generated by an element b under the operation $*$ if

$$S = \left\{b,\ (b * b),\ (b * b) * b,\ \left[(b * b) * b\right] * b,\ \ldots\right\}$$

Chapter 13

STATISTICS AND GRAPHS

13-1. Gathering Data.

If you look at the pupils in your classroom it may occur to you that several pupils appear to be about the same height, that some pupils are taller and some are shorter. Suppose you want to know the height of the tallest, the height of the shortest, and how many are the same height. How will you go about finding this information? You would first measure the height of each pupil. In this way you will be collecting facts to answer the questions you have in mind. Instead of collecting facts we may say you are gathering data. (The word "data" is the plural of the Latin word "datum" which means "fact".)

Let us suppose that you find each pupil's height correct to the nearest inch. After this is done, the measurements should be arranged in such a way that it is possible to select information from the data as easily as possible. Making such an arrangement is done frequently by putting the data in a table as shown in Table 13-1. If you wished, you might list the pupils by name, but here each pupil is assigned a number.

With the data arranged in this way it is very easy to answer such questions as the following. How tall is the tallest pupil? the shortest? How many pupils are 60 inches tall or taller? How many less than 60 inches tall? What height occurs most often?

Table 13-1

Heights of 15 Seventh Grade Pupils

Pupil	Height in Inches
1	65
2	64
3	63
4	61
5	61
6	61
7	60
8	59
9	57
10	55
11	55
12	54
13	54
14	53
15	52

This example about the heights of the pupils in a class is a sample of the kind of things we do in studying statistics. Statistics, in part at least, has to do with the collection of data and the making of tables and charts of numbers which represent the data. The tables and charts usually make it easier to understand the information which is contained in data that have been gathered. We will use the data in Table 13-1 later in this chapter to illustrate some other things that we do in our study of statistics.

Many of the duties of different agencies in the U. S. government could not be performed if the agencies were not able to collect a great many data to use in their work. The Congress of the United States has the power "to lay and collect taxes --- to pay the debts and provide for the common defense and general welfare of the United States." The amount of taxes to be collected depends on many things. Name some of them. Certainly one thing on which it depends is the number of people in the United States. The Congress must provide for counting the people "within every term of ten years." The census taken in 1950 showed that there were about 151,000,000 people in the United States. Another census was taken in 1960. Ask your librarian to help you find the population of the United States in 1960.

Table 13-2 shows the population in millions for every census from 1790 through 1950. The table shows that the population in 1790 was 3.9 millions. This means there were 3,900,000 (3.9 × 1,000,000) people in the U. S. at that time. Is this an exact or approximate number? The column which is headed Percent of Increase shows the percent of increase in the population during the preceding ten-year period.

Table 13-2

Population Facts About the United States

Census Years	Population in Millions	Increase in Millions	Percent of Increase
1790	3.9		
1800	5.3	1.4	35.1
1810	7.2	1.9	36.4
1820	9.6	2.4	33.1
1830	12.9	3.3	33.5
1840	17.1	4.2	32.7
1850	23.2	6.1	35.9
1860	31.4	8.2	35.6
1870	39.8	8.4	26.6
1880	50.2	10.4	26.0
1890	62.9	12.7	25.5
1900	76.0	13.1	20.7
1910	92.0	16.0	21.0
1920	105.7	13.7	14.9
1930	122.8	17.1	16.1
1940	131.7	8.9	7.2
1950	150.7	19.0	14.5

*From Statistical Abstract of the United States, 1956.

Exercises 13-1

1. Do you see any general trends in the data shown in the table?
2. In which decade was the percent of increase the largest? Do you know a reason for this from your study of history?
3. In which decade was the percent of increase the lowest? Can you explain this by history you have studied?

4. The Irish Famine occurred in the years 1845, 1846, 1847. How might this have affected the population of the United States?

5. What was the percent of increase in population from 1870 to 1880?

6. Arrange these data in a logical order. Students' test scores on a mathematics test were: 72%, 80%, 77%, 95%, 84%, 61%, 98%, 75%, 80%, 100%, 67%, 77%, 83%, 75%, 88%, 91%, 70%, 78%, 82%, 86%.

Use an almanac or other reference material to find the information needed in the problems that follow.

7. List the population of the largest city in your state for the years 1900, 1910, 1920, 1930, 1940 and 1950. (Use your city if it is large enough to be listed, or if you can get the figures.)

8. (a) List the number of immigrants to the U. S. for each year from 1935 to 1950.

   (b) Arrange the numbers in (a) from the smallest to the largest.

9. List the number of automobiles sold by the manufacturer in the U. S. in each of these years: 1910, 1915, 1920, 1925, 1930, 1935, 1940, 1945, 1950, 1955.

13-2. The Broken Line Graph.

Such data as we talked about in the last section are frequently represented by "drawing a picture." The "picture" for the data in Table 13-2 is shown in Figure 13-1. This "picture" is called a broken-line graph, as many of you already know. Such graphs are usually made on cross-section paper like that used in Figure 13-1. Broken line graphs are used to show change in some item.

POPULATION OF THE UNITED STATES

1790 - 1960

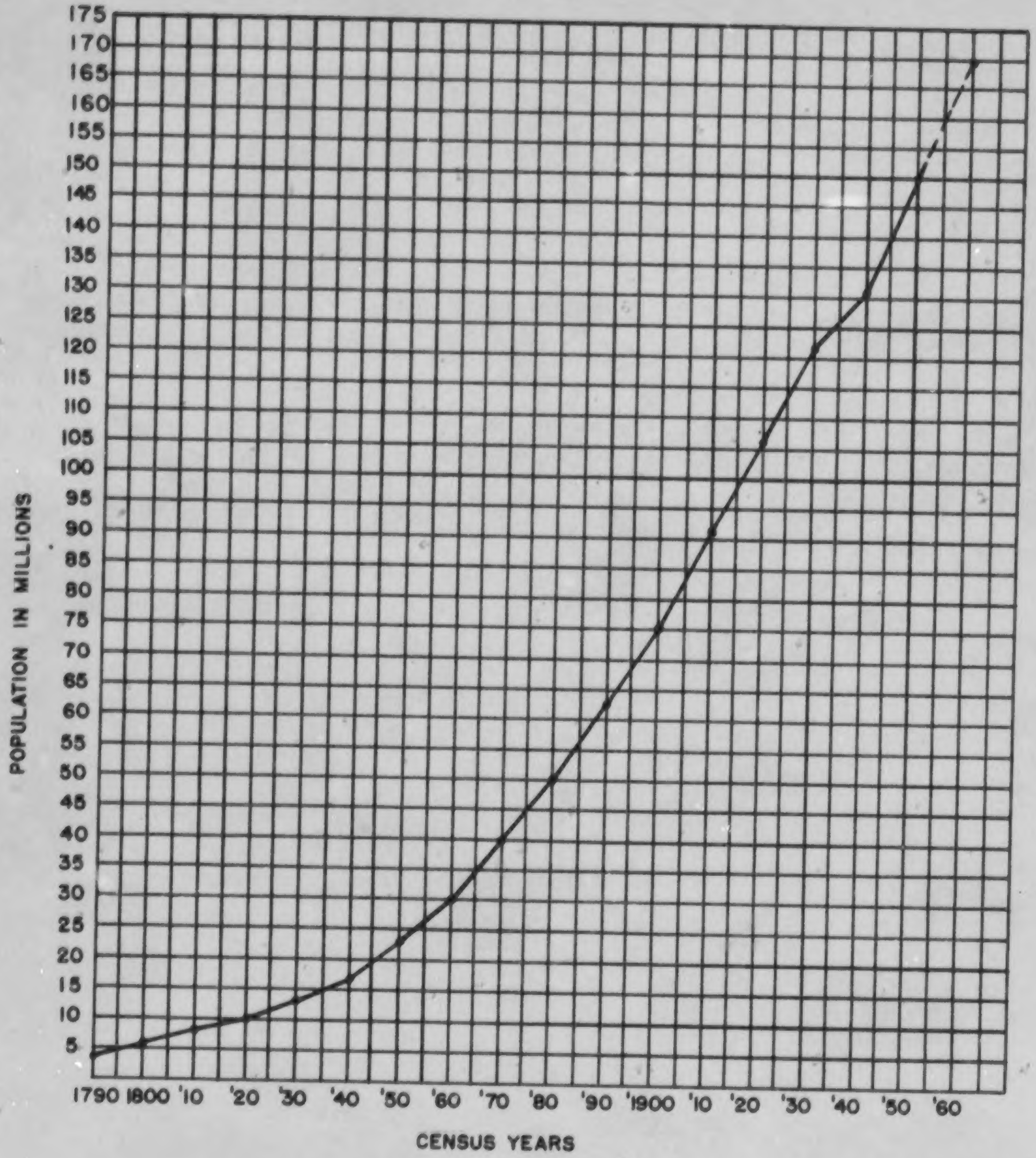


Figure 13-1

One horizontal line and one vertical line are used in such a way that they resemble a number line. In Figure 13-1, the horizontal line is used to show time. Notice that each ten year period is represented by the same distance along the line. The vertical line shows the number of people. How many people does each unit represent? These two lines are the reference lines of the graph. They show the scales used to draw the graph.

Would the numbers on these scales have any meaning if the scales were not named? In fact, would the graph mean much if it did not have a title? A good graph must be neat and "eye-catching," have a title that identifies the information clearly, and scales that are meaningful and easy to read.

Each point on the graph in Figure 13-1 represents the population for the year whose number on the horizontal scale is directly below the point. The population for each point is read by drawing a perpendicular from the point to the vertical scale and reading the number there. Numbers used on graphs are usually approximate numbers so the number read for any point is also an approximate number. To make the graph, the points that represent the population for each census year are plotted; then they are connected by line segments. The segments between points give estimates of the population between census years. There is no assurance that these estimates are accurate, however, since the change may have been more rapid at one time than at another.

Class Discussion Questions

Use Figure 13-1 to answer the questions.

1. Did the population increase more between 1900 and 1910 than between 1800 and 1810?
2. Does the graph show a decrease in population over any ten-year period?
3. If the population for 1810 and 1820 had been the same how would this be shown in the graph?

4. What was the approximate population in 1945? in 1895? How much had it changed in the 50 years between these two dates?

5. If the population increases at the same rate from 1950 to 1960 as from 1940 to 1950 (if the graph climbs in a straight line from 1940 to 1960), what should the population be in 1960? This portion of the graph is indicated by a dotted line. If the census for 1960 is known at the time you answer the question, compare the census and your answer.

Making Line Graphs

When drawing graphs, the details should be planned before any marks are put on paper. Look at the information you wish to show and the space that is available. Leave plenty of room for a title and for labeling scales. Graphs look better if all words are printed rather than written in long-hand. Graphs should be as large as space permits. Rulers make lines straight, and line segments enclosing the graph give the graph a finished appearance.

Line graphs require the use of scales. The biggest problem in making a graph is deciding on the scale. There are ways to tell how much each unit should represent. When graph paper is used, the squares of the paper mark convenient units. As an example, we will use the data in Table 13-2. Count the number of census years from 1790 to 1960. Since there are 18, the graph must have at least 17 units along the horizontal scale. (Why are only 17 needed?) The horizontal scale always starts at the left edge of the graph and a number is placed there. If 344 units were available for use on the horizontal scale, what could you do? Count the squares along the horizontal scale in Figure 13-1. Two squares were used for each census year in the graph.

Next, look at the vertical scale. It shows the number of millions of people in the United States. From Table 13-2 find the largest number of people that needs to be shown on the graph. The numbers in this table should be rounded to the nearest million before you work with them. Count the number of units from the horizontal scale to the top of the space used. (Leave room at the top for a title.) In Figure 13-1 there are 35 units available along the vertical axis, and the largest number of people is 151,000,000. Divide the number of people by the number of units available. In this example the quotient comes out with the atrocious number 4,314,285 and there is a remainder. According to this, each unit should represent 4,314,286 people; it would be very difficult to work with such a number. Each unit of the vertical scale should represent the next larger, convenient number. In this case, each unit should represent 5,000,000 people. Why must the number used for each unit be larger than the quotient?

Is it necessary to do the entire division shown above? Since we will use approximate numbers for the graph, the division may also be approximate. The population was rounded to the nearest million. The division needs to be carried out only until the quotient shows the number of million. Now the problem is easy.

$$\begin{array}{rl} & 4 \\ 35 & \overline{)151,000,000} \\ & \underline{140} \\ & 11 \end{array}$$

The rest of the figures make no difference, since the number used as a unit must be larger than the quotient.

Graphs are usually neater if only enough lines of the scales are named so as to make the scale easily readable.

Exercises 13-2

1. Find the number that each unit should represent for each of the following situations.

	Number of squares available	Largest number to be graphed
(a)	20	400
(b)	20	475
(c)	20	175,000
(d)	33	940
(e)	27	2,465,100

2. Make a broken line graph to represent the data given in this table:

The number of students in Franklin Junior High School for years 1952-1957:

1952 --- 86
1953 --- 150
1954 --- 164
1955 --- 196
1956 --- 235
1957 --- 254

3. Make a broken line graph to represent the data in Table 13-3.

Table 13-3

Popular Vote in Millions Cast for Presidential Candidates of the U. S. 1928 to 1956

Year	Republican Party	Democratic Party
1928	21.4	15.0
1932	15.8	22.8
1936	16.7	27.5
1940	22.3	26.8
1944	22.0	24.8
1948	22.0	24.1
1952	33.8	27.3
1956	35.6	25.7

Observe these instructions:

(a) In the same graph draw one broken line for the Republican party and one for the Democratic party.

(b) Find in your textbook, or elsewhere, the name of the president elected and the name of the unsuccessful candidate in each election.

(c) Use Tables 13-2 and 13-3 to find the total percent of the population who voted for either the Republican or Democratic party candidate in the presidential election in 1940.

4. Make a line graph for the data for Problem 7, Exercise 13-1.

5. Make a line graph for the data for Problem 8, Exercise 13-1.

6. Make a line graph for the data for Problem 9, Exercise 13-1.

13-3. Bar Graphs.

Table 13-4 gives the pupil enrollment in the seventh grade in the U. S. for the years 1952-1959 and the expected enrollment for the years 1960-1962.

Table 13-4

Seventh Grade Pupil Enrollment in U. S. 1952-1962

Year	Enrollment in thousands
1952	2,159
1953	2,224
1954	2,354
1955	2,521
1956	2,586
1957	2,599
1958	2,707
1959	3,075
*1960	3,260
*1961	3,302
*1962	3,333

*Expected enrollment for the years 1960-1962.

The data in this table are represented by the graph in Figure 13-2. This kind of graph is called a bar graph. Bar graphs show comparison between similar items. Data suitable for line graphs may also be suitable for bar graphs. Data that show change may also be considered as comparing similar sets; the number associated with each time period is then considered as a set of similar items. There are some sets of data that are suitable for bar graphs that are not good material for line graphs. An example would be a graph comparing the heights of the 10 highest mountains in North America. In Figure 13-2 the years are represented along the horizontal base line and the enrollment in thousands is represented along the vertical line at the left. The bars are spaced along the base line so that the distance between any two bars is the same. The width of each bar is the same also. In this graph, the width of the spaces and the bars are the same, but this does not have to be true. The name of each bar is the number of the year printed at the bottom of each bar. In some graphs, the bars will have names that are not numbers. When planning a graph, be sure to leave enough room to print the name of the bar.

The number represented by each bar can be read from the vertical scale. It is the number represented by the point on the vertical scale which is on the same horizontal line as the top of the bar.

In making bar graphs, be sure to observe all principles for constructing good graphs. Plan to use the entire width of available space. Since the width for spaces may differ from the width for bars, this is possible. The number that each unit on the vertical scale represents is found in the same way as for line graphs.

SEVENTH GRADE PUPIL ENROLLMENT IN U. S.
1952 - 1962

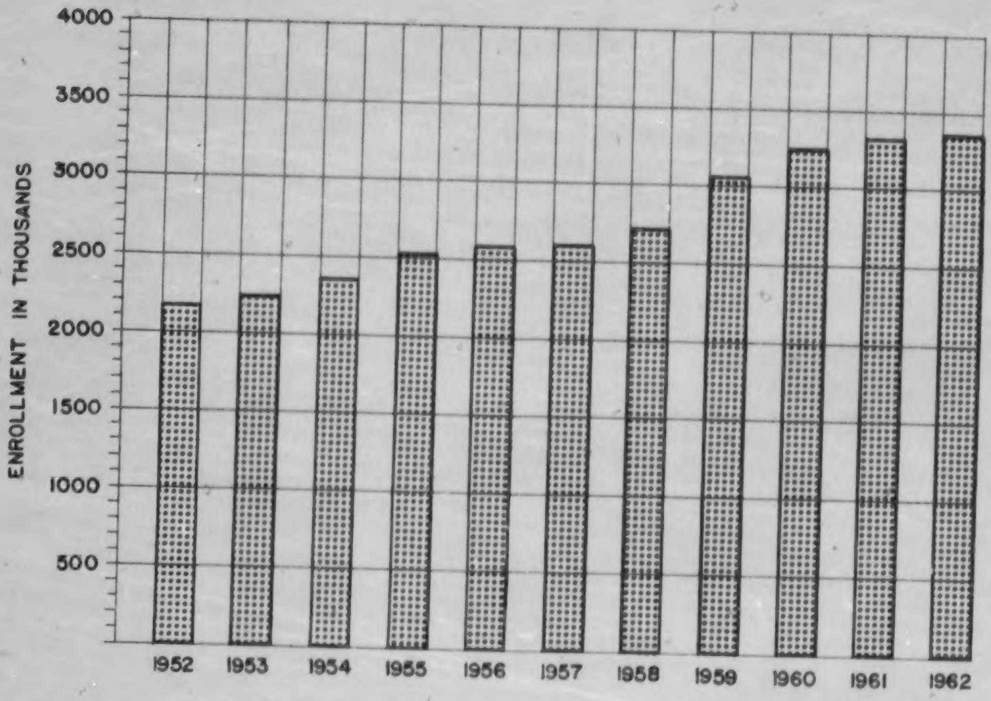


Figure 13-2

Exercises 13-3

1. If each seventh grade pupil in 1960 needs a mathematics textbook which costs $3.25 what will be the total amount spent for books to supply the whole seventh grade?

2. During which year was the enrollment about 3 million pupils?

3. Between which two years did the greatest change in enrollment occur? Use Figure 13-2 to obtain the answer, then use Table 13-4 to see if your answer is correct.

4. Draw a bar graph to represent the number of people killed in different types of accidents during 1956 as shown in this table:

Motor vehicle accidents	40,000
Falls	20,200
Fires and injuries from fires	6,500
Drownings	6,100
Railroad accidents	2,650

5. The highest altitude in each of the states listed is given. Round the data to the nearest 100 feet, then show in a bar graph.

Alabama	2,407 ft.
Alaska	20,320 ft.
Arizona	12,670 ft.
Arkansas	2,830 ft.
California	14,495 ft.
Colorado	14,431 ft.

6. At one time during the 1960 baseball season, the National League teams had won the number of games shown. Show this in a bar graph.

Pittsburgh	64	San Francisco	51
St. Louis	60	Cincinnati	46
Milwaukee	57	Philadelphia	42
Los Angeles	56	Chicago	39

13-4. Circle Graphs.

A circle graph is shown in Figure 13-3. Such a graph is used to show the comparison between the parts of a whole and between the whole and any of its parts. This graph shows the percent of income which a family spent for food, clothing, rent and miscellaneous expenses and the percent of income which was put into savings.

HOW ONE FAMILY SPENDS ITS MONEY

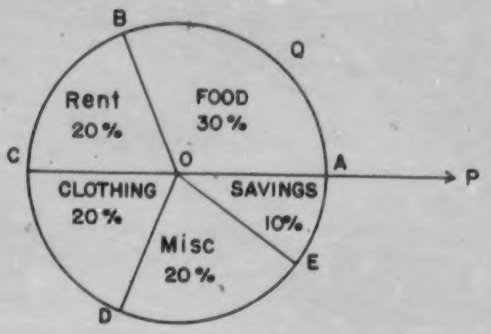


Figure 13-3

The family spent 30 percent of its income for food, 20 percent for clothing, 20 percent for rent, 20 percent for miscellaneous expenses, and saved 10 percent. The family's total income is represented by the area of the circle. Any length can be chosen for the length of the radius. The length is usually chosen so that the circle is large enough to show clearly the parts into which it is divided and small enough to get it in the available space.

Since the part of the income spent for food is 30 percent we must have an area which is 30 percent of the circle's area to represent this part of the income. To get this area we begin by drawing the ray $\overrightarrow{OP}$. Label with A the intersection point of the ray and the circle. We know that we can divide the circle into 360 equal parts by drawing 360 angles of 1 degree, each with its vertex at 0. Thirty percent of 360° is 108°. If the protractor is laid along $\overrightarrow{OA}$, with the vertex mark at 0 and the 0° mark on OA, then the 108° mark will be on the ray $\overrightarrow{OB}$.

The area bounded by the closed curve OAQB is 30 percent of the area of the circle. Hence the interior of the closed curve OAQB represents the part of the income spent for food.

What percent of the circle's area represents rent? Twenty percent of 360° is 72°. By placing your protractor along $\overrightarrow{OB}$ with the vertex mark at O and the 0° mark on $\overrightarrow{OB}$, then the 72° mark will be on the ray $\overrightarrow{OC}$. Continue in the same manner to check the area representing clothing, miscellaneous, and savings.

If the family's income is $6,000 how much is spent for food? How much is put into savings?

In a certain school there are 480 pupils. At lunch time 80 pupils go home for lunch, 120 bring their lunch, and 280 buy their lunch in the school lunchroom. The circle graph in Figure 13-4 shows the way the pupils are divided at lunch time. Before making this circle graph we had to find either what percent or what fractional part of the pupils go home for lunch, bring their lunch, and buy their lunch. These fractions and percents are shown in the table.

Number of Pupils		Fractional Part of Total	Percent	Degrees
Go home	80	$\frac{1}{6}$	$16\frac{2}{3}$	60
Bring lunch	120	$\frac{1}{4}$	25	90
Buy lunch	280	$\frac{7}{12}$	$58\frac{1}{3}$	210
Total	480	1	100	360

SOURCES OF PUPILS' LUNCHES

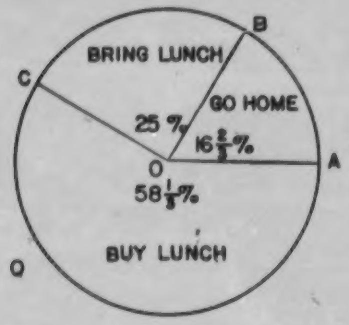


Figure 13-4

Now we find the number of degrees in the curve AB, by taking $\frac{1}{6}$ of 360 to get 60; the number of degrees in the curve BC by taking $\frac{1}{4}$ of 360 to get 90; and the number of degrees in the curve CQA by taking $\frac{7}{12}$ of 360 to get 210.

Exercises 13-4

Make a circle graph to represent the information in each of the following problems. Make one graph for each problem. Round to nearest degree.

1. In 1949 it was found that school-related accidents which involved seventh grade pupils in the U. S. happened as shown. (Figures given are very close approximations to actual figures.)

   60 percent of the accidents happened in the school buildings.
   30 percent of the accidents happened on the school playgrounds.
   10 percent of the accidents happened on the way to and from school.

2. In 1956 the figures for the accidents of Problem 1 were as follows: (Again, figures are very close approximations.)

   36 percent of the accidents happened in the school buildings.

   54 percent of the accidents happened on the school playground.

   10 percent of the accidents happened on the way to and from school.

3. The Drama Club earned money during the year to buy new curtains for the clubroom stage. About 79% of the cost of the curtains was earned by selling tickets to a series of school plays. Approximately 16% of the money needed came from selling programs, and the remaining 5% came from a candy sale.

4. At Washington Jr. High School 50% of the pupils live a mile or less from school. Approximately 28% of them live more than a mile but less than two miles from school, while 22% come from homes two miles or more away.

13-5. Summarizing Data.

Some information can be determined easily by looking at all the data in table form. Sometimes, however, a large number of items in a table makes it confusing. In this case it may be better to describe the data by using only a few numbers. Finding an average of such a set of numbers is often very helpful in studying the data given to you.

Do you know how to find an average? You have been doing this for some time, but did you know there are several kinds of averages?

Arithmetic Mean

When you calculate the average from a set of numerical grades by adding the numerical grades and dividing the sum by the number of grades, you find a number which you use as a representative of the numbers in the set. This useful average, with which you are already familiar, is called arithmetic mean or the mean. (When the word mean is used in this chapter it always refers to the arithmetic mean.)

Let us look once more at the heights recorded in Table 13-1, which is reprinted below. This table gives a list of the heights, ordered from largest to smallest, of 15 pupils.

Table 13-1

Heights of 15 Seventh Grade Pupils

Pupil	Height in Inches
1	65
2	64
3	63
4	61
5	61
6	61
7	60
8	59
9	57
10	55
11	55
12	54
13	54
14	53
15	52

In describing this set of data, can we find one number which we can use to represent the numbers in these measurements? One such number would be the arithmetic mean. For this table the average height (arithmetic mean) is $\frac{\text{sum of the heights}}{\text{number of pupils}} = \frac{874}{15} \approx 58$. This commonly used measure can be found without arranging the data in any special way.

Median

Another way to obtain one number representing the numbers in a set of data is to find a number so that half of the numbers in the set are greater and half are less than the number found.

The median of a set of numbers is the middle one of the set when the numbers in the set are arranged in order, either from smallest to largest or from largest to smallest. In the set of heights in Table 13-1 the middle number is 59. This is the median of the set. Half of the numbers are greater than 59 and half are less. Seven pupils are taller than 59 inches and seven are shorter than 59 inches.

If the number of elements in the set is even, there is no middle number. Thus we must define the median for this situation. If there is an even number of elements in the set, the median is commonly taken as the mean of the two middle numbers. For example, in the set of numbers 8, 10, 11, 12, 14, 16, 17, 19 the two middle numbers are 12 and 14. The median is 13, the mean of 12 and 14, although 13 is not in the set. Sometimes several items are the same as the median. The set of scores 12, 13, 15, 15, 15, 15, 16, 18, 19, 20 has 10 numbers in it. The two middle ones are 15 and hence the median is 15. But the third and fourth scores are also 15 so that 15 is not a score such that 5 scores are smaller and 5 are larger than it.

In the set of salaries \$2050, \$2100, \$2300, \$2400, \$2500, \$2600, \$2700, \$2700, \$2700, \$3150 the median salary is \$2550. The arithmetic mean is \$2520. The median and arithmetic mean are nearly equal. But, if the largest salary had been \$5150 instead of \$3150 the arithmetic mean would have been \$2720 and the median would have still been \$2550. This illustrates that the usefulness of the median in describing a set of numbers often lies in the fact that one number (or a few numbers in the set) does not affect the median as it does affect the arithmetic mean.

Mode

Which height occurs more than any other in Table 13-1? How many pupils have this height? This height is called the mode.

In sets such as the natural numbers 1, 2, 3, 4, 5, ... no number occurs in the set more than once. But in a set of data some number, or numbers, may occur more than once. If a number occurs in the set of data more often than any other number it is called the mode. (We might say it is the most fashionable.)

There may be several modes. In Table 13-1 there was just one, 61. In the set of salaries $2,050, $2,100, $2,300, $2,400, $2,500, $2,600, $2,700, $2,700, $3,150 the mode is $2700. But in the set of scores 19, 20, 21, 21, 21, 24, 26, 26, 26, 29, 30 there are two modes 21 and 26. (These are equally fashionable.) If there had been another score of 21 in this set of scores, what would the mode have been? In Table 13-1 if the 12th pupil were 55 inches tall how would this affect the mode?

Exercises 13-5a

1. Find the mode of the following list of chapter test scores: 79, 94, 85, 81, 74, 85, 91, 87, 69, 85, 83.
2. From the scores in Problem 1, find the:

   (a) Mean (b) Median
3. The following annual salaries were received by a group of ten employees:

   $4,000, $6,000, $12,500, $5,000, $7,000, $5,500, $4,500, $5,000, $6,500, $5,000.

   (a) Find the mean of the data.

   (b) How many salaries are greater than the mean?

   (c) How many salaries are less than the mean?

   (d) Does the mean seem to be a fair way to describe the typical salary for these employees?

   (e) Find the median of the set of data.

   (f) Does the median seem to be a fair average to use for this data?
4. Following are the temperatures in degrees Fahrenheit at 6 p.m. for a two week period in a certain city:

   47, 68, 58, 80, 42, 43, 68, 74, 43, 46, 48, 76, 48, 50

   Find the (a) Mean (b) Median

Grouping Data

If you were listing heights of a very large group of pupils, it might be inconvenient to list each one separately. It would be simpler to group the figures in some such way as this:

Height in Inches	Number of pupils
62-64	12
59-61	17
56-58	42
53-55	57
50-52	33
47-49	14

In order to find the median first find the total number of pupils and divide by 2. The sum of 12 + 17 + 42 + 57 + 33 + 14 is 175, and $\frac{175}{2} = 87\frac{1}{2}$ so the middle person will be the 88th one, counting from the top or bottom. If we count down from the top, 12 + 17 + 42 = 71. We need 17 more to reach 88. Counting down 17 more in the group of 57 brings us to the median. Since the 88th person is within that group, we say that the median height of the whole group of pupils is between 53 and 55 inches. Since the 88th person comes before we reach the middle of that group as we count down, we might say that the median height is likely to be nearer 55 than 53.

Let's check our work and count up from the bottom to the 88th person. 14 + 33 = 47. We need 41 more than 47 to make 88, so we count 41 more and that takes us into the upper part of the group of 57 as we found when we counted down from the top. Again you find the 88th person in the group of 57 whose height is between 53 and 55. Thus the median height of the group is between 53 and 55 inches.

Exercises 13-5b

1. Give an example in which your principal might choose to group data rather than use individual numbers.

2. Find the median of the following age groups. What is the median age?.

Ages in Years	Number in Group
27-29	35
24-26	48
21-23	68
18-20	18
15-17	94
12-14	53
9-11	73
6-8	26

3. Find the median by grouping the following data on temperatures: (Use intervals of 5, namely 50-54, 55-59, etc., to 90-94.) 62, 74, 73, 91, 68, 84, 75, 76, 80, 77, 68, 72, 71, 86, 82, 74, 55, 72, 50, 63, 71. Label one column "Temperature" and the other column "Frequency".

Scatter

The mean, the mode, and the median are averages. Each is a measure which gives us an idea of the size of the measurements. Each may be thought of as a typical or representative number.

If we want to summarize a set of data we can do this with just two numbers. One of the numbers will be an average (mean, mode, or median) to give us a notion of the size of a typical element. The other number in our summary will give us a measure of how the data is scattered from our average. Suppose we wish to summarize the following two sets of numbers:

$$A = \{40, \quad 50, \quad 60\}$$
$$B = \{49, \quad 50, \quad 51\}$$

The mean and the median of set A is the same as the mean and median of set B but the numbers in set B are spread farther apart than the numbers in set A. One way to measure this "spread" or "scatter" is to find the difference between the largest and the smallest number in the set. This difference is called the range. The range for the numbers in set A is 20;

the range for the numbers in set B is 2.

We can now describe set A as a set of numbers whose mean is 50 and whose range is 20. These two numbers give us a short description of the set. We can describe the heights of the seventh grade pupils given in Table 13-1 by saying that the mean height is 58 inches and the range is 13 inches. From this we can think that a measure of the heights is about 58 and that a measure of the spread of the heights is 12.

Another measure of the spread or scatter of a set of numbers is the *average deviation*. To find the average deviation we must first find the *deviation* or difference of each number from the arithmetic mean.

Consider the set of numbers 4, 8, 10, 4, 5, 4, 7. What is the mean of this set? What is the difference (deviation) of the largest number in this set from the mean? The deviations of the numbers in this set from the mean are 2, 2, 4, 2, 1, 2, 1. These numbers tell us how the numbers scatter, or deviate, from the mean. The average deviation is the mean of these deviations. Taking the mean of 2, 2, 4, 2, 1, 2, 1 we get $\frac{2+2+4+2+1+2+1}{7} = 2$. Let us use this measure, the average deviation, in describing another set of data.

The total receipts of the Federal Government in the years 1946-1955 were as follows:

Year	Billions	Deviations from Mean
1946	44	11.5
1947	45	10.5
1948	46	9.5
1949	43	12.5
1950	41	14.5
1951	53	2.5
1952	68	12.5
1953	73	17.5
1954	73	17.5
1955	69	13.5
Total	555	122.0

The arithmetic mean of these receipts is the total, 555, divided by 10, or 55.5. (555 and 55.5 are numbers of billions.)

The third column shows the deviation of each year's receipts from the mean, 55.5.

The mean of the deviations is 122 divided by 10, or 12.2

We can now condense the information in the table by saying: The receipts of the Federal Government in the years 1946-1955, had a mean of 55.5 billion dollars and an average deviation of 12.2 billion dollars.

Exercises 13-5c

(The first three questions in these Exercises refer to the data for the receipts of the federal government in 1946-1955.)

1. In which year(s) was the deviation from the mean the greatest?
2. In which year(s) was the deviation from the mean the least?
3. Find the mean for the years 1946-1949 and the average deviation from the mean.
4. Find, to the nearest tenth, the mean and the average deviation of the following test scores:

   85, 82, 88, 76, 90, 84, 80, 82, 84, 83.
5. Find the mean and the average deviation of the test scores (same test, but in another class):

   94, 84, 68, 74, 98, 70, 96, 84, 76, 96.

*6. Another method of calculating the arithmetic mean is shown in the following example.

Example. Calculate the arithmetic mean of the set of stores:
10, 11, 13, 15, 19, 20, 21, 21, 23

We begin by making a reasonable guess of some number for the mean. Suppose we guess 18. Next we find the deviation of each score in the set from 18.

Scores	Deviations from 18
10	8
11	7
13	5
15	3
19	1
20	2
21	3
21	3
23	5

The sum of the deviations for the scores less than 18 is 23. The sum of the deviations for the scores greater than 18 is 14. We take the difference 23 - 14 = 9 and divide this by 9, the number of scores: 9 ÷ 9 = 1. Since the deviations from 18 were greater for the scores less than 18 than for the scores greater than 18 we subtract 1 from 18 to get the correct mean of 17.

Use the method of the above example to find the mean of the set of scores: 40, 43, 44, 47, 48, 49, 51.

The mean, median, and mode, for a set of data are called measures of central tendency. This name is given to them since each of them is a number about which the data tends to "center." All of these measures of central tendency are illustrated in the following set of salaries of 12 people and in the graph in Figure 13-5. Salaries: \$4,000, \$4,500, \$4,500, \$5,000, \$5,000, \$5,000, \$5,250, \$5,250, \$5,250, \$5,250, \$5,500, \$6,000.

SALARIES OF A SELECTED GROUP

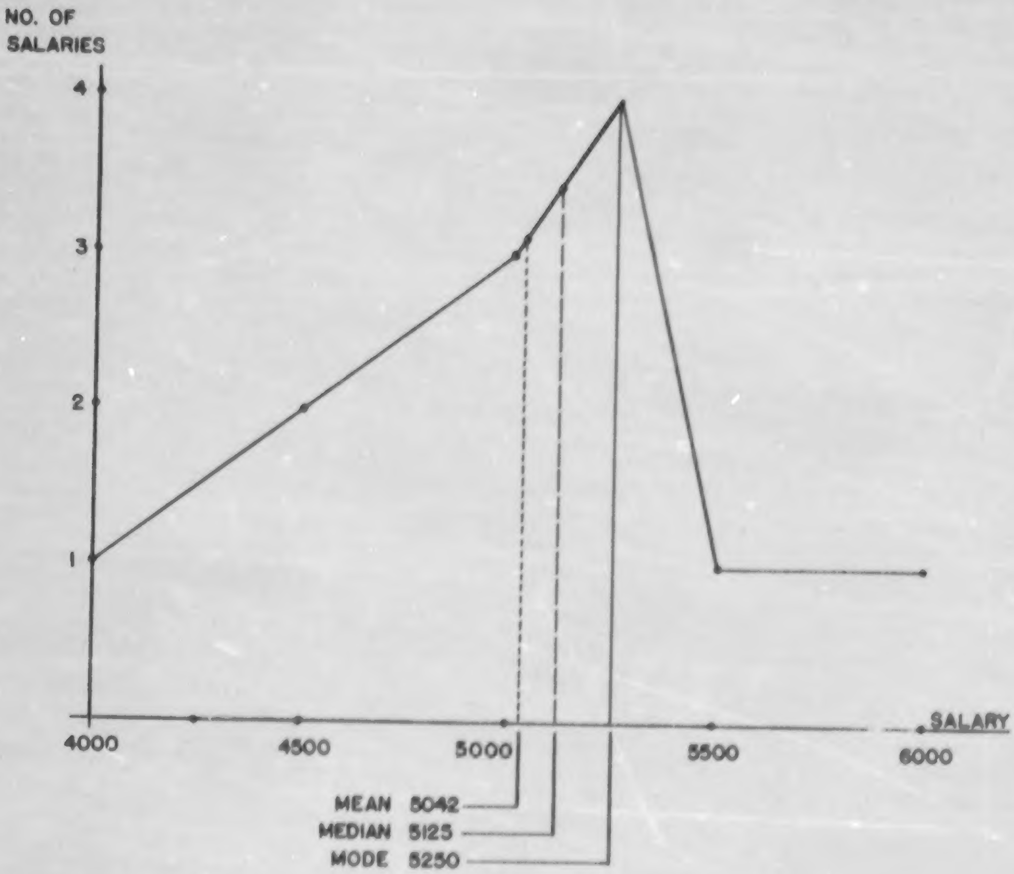


Figure 13-5

Calculation of average deviation from the mean, $5,042 (to the nearest dollar).

Salaries	Deviation from $5,042
$4,000	$1042
4,500	542
4,500	542
5,000	42
5,000	42
5,000	42
5,250	208
5,250	208
5,250	208
5,250	208
5,500	458
6,000	958
Total	$4500

$\frac{\$4500}{12}$ = $375, average deviation from the mean.

Range: $6000 - $4000 = $2000.

The locations of the lines which represent the mean, median, and mode show that these numbers are nearly equal and the graph shows the salaries are about equally distributed on both sides of these lines.

13-6. Sampling.

We all know that a presidential election is held every four years in the United States. In which year will the next one be held? People are very much interested in the outcome of the elections. Sometimes, long before the elections are held, organizations make predictions concerning who will be elected. These organizations not only predict who will be elected but even predict the percent of the votes cast that each candidate will receive. The candidates and the percent of vote predicted for each of them in the election of 1948 by three different polls is shown in the table.

Candidates	Dewey	Truman	Thurmond	Wallace
Poll No. 1	49.5%	44.5%	2%	4%
Poll No. 2	49.9%	44.8%	1.6%	3.3%
Poll No. 3	52.2%	37.1%	5.2%	4.3%

In the election the percent of vote for each was: Truman 49.4%, Dewey 45.0%, Thurmond 2.4%, Wallace 2.4%. Do you see why this election is called the "surprise election"?

Although none of the polls predicted the election correctly, their predictions were close. How did they do it? Did they go about the U.S. and ask every voter how he was going to vote? Or, did they write each voter a letter? Either of these ways would have been very expensive and would have taken a great deal of time. Instead of either of these they used a method called sampling. This means that the organizations who made the predictions selected a "sample" of the population of the U.S. Then, after asking the people in the "sample" how they would vote, the organizations predicted that the vote in the entire country would be in nearly the same ratio as the vote in the "sample."

If you have ever had a blood-count, the doctor took a very small amount of blood from the tip of your finger, or from your ear lobe, and then counted the red and white corpuscles in it. This was a very small sample of your blood. The count in it was taken as a reliable representation of the count in all the blood in your body. Perhaps you can think of other examples of sampling.

Let us suppose you know that all the employees whose names are listed in the employee's directory of a certain large manufacturing firm are men over twenty-one years of age. Then let us ask how we might use the sampling method to make an estimate of the average height of these men. You might select the first and last man listed in the directory and find their average height. Or, you might select the first name listed under each letter of the alphabet, or the last name under each letter, or both the first and last names under each letter. There are many ways a sample could be selected. Some would be good and some would be bad. Do you see any objection to any of the methods suggested? The way of selecting a sample so that it will be a good representation of the group from whom the sample is selected is a very difficult part of the job.

Exercises 13-6

1. This problem is a sampling project for your class. The objective is to find the mean height of your classmates. Since there is likely to be a difference in boys' heights and girls' heights, find the mean for boys and girls separately. Boys should find the boys' mean and girls should find the girls' mean. All directions are given for boys; girls should substitute the word "girls."

    (a) Select a sample in each of these ways.

        1. All boys whose birthdays are in March, August or December.

        2. All boys whose first names start with G, M, or T.

        3. All boys who sit in some one row in the mathematics classroom.

    (b) Find the mean height using Sample 1.

    (c) Find the mean height using Sample 2.

    (d) Find the mean height using Sample 3.

    (e) Find the mean height of all boys in the class.

    (f) Which sample was the best representation of the heights of all the boys in the class?

    (g) In selecting samples, is it important that the sample be chosen so as not to give one part too much representation?

2. There were 48,834,000 (to the nearest thousand) votes cast in the 1948 election for President of the United States.

   (a) If Poll No. 1 on page 505 of the text had been correct, how many votes would have been cast for each candidate? Give your answer to the nearest 10,000.

   (b) The percent of the vote that was received by each candidate is given under the table of the polls. Use these percents to find the number of votes received by each candidate. Round this to the nearest 10,000 votes.

13-7. Summary.

The subject matter of statistics deals, in part, with collecting data, putting it in table form, and representing it by graphs. The tabulating and graphing of the data should be done in ways such that the story told by the data can be interpreted and summarized easily. The broken line graphs, bar graphs, and circle graphs are just a few of the kinds of graphs that may be used.

You have learned that there are many different measures for the central tendency of the same set of data. The next time you see graphs or tables of a set of figures in the magazines, newspapers, or your social studies book, look them over carefully. If averages are mentioned, see if you can tell which average is used. Whatever kind of average is used, you have the right to question whether the average used gives the best representation of all the data.

To help you recall the new terms you have used in working with statistics, they are listed for you:

Arithmetic mean or mean -- the sum of all the numbers in a set divided by the number of members in the set.

Median -- the middle number when data is ordered either from smallest to largest or largest to smallest. When there is no one middle number, the average of the two middle numbers is the median.

Mode -- the number occurring most in the list of data. There may be several modes.

Range -- difference between largest and smallest number in a set.

Average deviation -- average of the deviations from the mean.

Chapter 14

MATHEMATICS AT WORK IN SCIENCE

14-1. The Scientific Seesaw.

Have you ever played on a seesaw?

If you weigh 100 pounds and your partner on the other side of the seesaw weighs 85 pounds, where does he have to sit to make the two sides balance? Will he be closer to the center or farther from it than you? Can you tell how far?

The seesaw is one form of simple machine that is used a great deal around home, at work, and in science laboratories. It belongs to the family of machines called "levers." Of course, not all levers work like seesaws! You use a lever to open a soft drink bottle. You use other forms of levers to jack up your car or to pry up a stone. Can you think of other examples of levers?

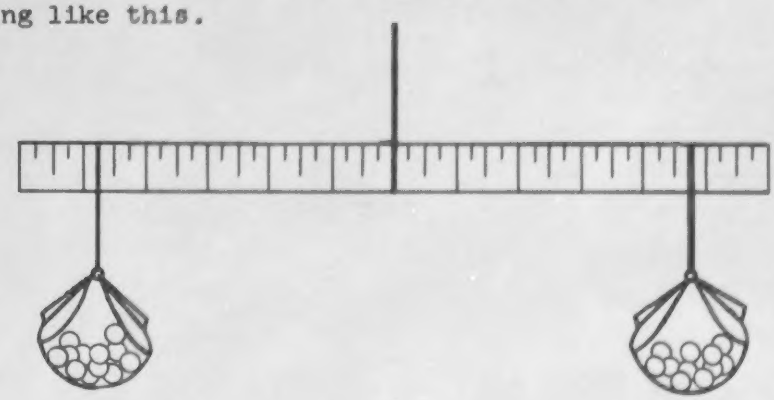
The scientist uses levers of rather fine construction for many purposes in his laboratory. The simplest type is the common laboratory balance or scale. You probably have one in your science room. Scientists long ago studied these scientific seesaws, learned how to balance objects of different weights, and expressed their findings in a mathematical formula. Having the formula makes the scientific seesaw much easier to use and understand.

Today you are going to play the part of a scientist. You will set up a simple seesaw experiment, make observations, try to discover a rule and try to state the rule in mathematical form.

The experiment suggests how a scientist makes observations in the laboratory, studies them mathematically and draws conclusions from them. He then tries to state the conclusions by means of a mathematical equation. Finally he uses the formula to predict a new result and then goes back to the laboratory to test whether the rule works in similar situations.

14-2. A Laboratory Experiment.

Your equipment to study the scientific seesaw will look something like this.



The materials required in your laboratory are:

A meter stick or yardstick.
Strong thread and two bags to hold the weights (thin plastic makes very satisfactory bags).
A set of objects of equal weight and convenient size (a supply of pennies or marbles is recommended).

Procedure:

1. Balance the stick by suspending it from a strong thread tied at the middle of the stick. If the stick does not balance, a thumbtack may be placed at various points until the stick is in balance. This point at which the stick is suspended is called the fulcrum.

2. Hang ten objects of equal weight on one side of the fulcrum and ten identical objects on the other side and try to make them balance. Take two objects of unequal weight and try to make them balance. Do you find that you must change the distance to achieve a balance when the weights are unequal?

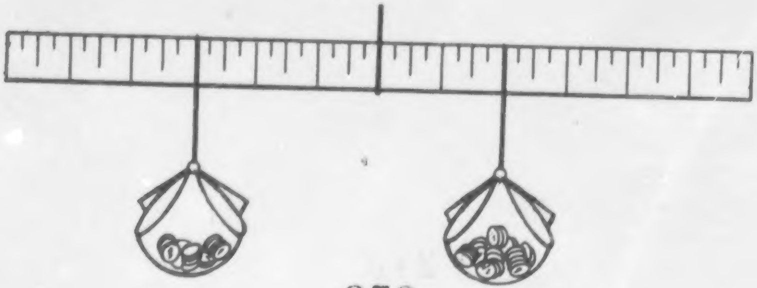
Note: Scientists usually perform some preliminary tests to determine the best way to set up and carry out the experiment. Their first experimental set-up doesn't always work perfectly! You may find it advisable to make some improvements in your equipment and procedures at this stage.

When you have your equipment operating smoothly, you are ready to take the first steps in your experiment. Scientists usually have the experiment carefully planned in advance, but we shall develop our plan as we go along.

3. (a) Hang 10 pennies or marbles (or other handy items of equal weight) at a distance of 12 centimeters from the fulcrum, and balance it with 10 identical objects on the other side. (On a yardstick you may find that $\frac{1}{2}$ inch is a convenient unit of distance.) Observe the distance of this second mass from the fulcrum when the lever is in balance and record the distance in a table, column (a), similar to the one below. Note that w and d represent the measures of the weight and distance respectively on one side of the fulcrum. W and D represent the measures of the weight and distance on the other side of the fulcrum.

TABLE I

	(a)	(b)	(c)	(d)	(e)	(f)	(g)	(h)	(i)	(j)
w =	10	10	10	10	10	10	10	10	10	10
d =	12	12	12	12	12	12	12	12	12	12
W =	10	20	5	8	15	24	12			
D =										



(b) If W is 20 find D so that the lever is in balance. Write the number in your table under "20," column (b).

(c) If W is 5 find D so that the lever is in balance. Write it in the table, column (c) under "5."

(d) Notice that in these first three trials, w and d remained the same. All changes were made in W and D. Use the values indicated in columns (d) - (g) and find the value of D. Make several other changes for W and write the corresponding results for D in your table -- columns (h), (i), and (j).

4. Now, as indicated in Table II, let w be 16 and d be 6. Find how large W will have to be for the lever to balance when D equals 6. How great a weight will balance the lever 4 cm. from the fulcrum? 16 cm? Try several other distances from the fulcrum, find what weight is needed to balance the lever, and fill in your Table II.

TABLE II

w	16	16	16	16	16	16	16
d	6	6	6	6	6	6	6
W							
D	6	4	16				

5. Try other values for the weights and distances as suggested in Tables III and IV and fill in similar tables of your own.

TABLE III

w	20	40	10			
d						
W	20	20	20	20	20	20
D	15	15	15	15	15	15

TABLE IV

w	18	18	18	18	18	18	18
d	5	5	5	5	5	5	5
W	15	10	30	45			
D					10	15	18

14-3. Caution: Inductive Reasoning at Work!

After a scientist has completed an experiment and collected data, he tries to analyze it. He tries to discover all the facts that the data present. He attempts to interpret these facts in a connected and convenient way and to discover a general rule which the data suggest. His hope is to state these results in a precise way, preferably by a simple mathematical formula.

Note that the scientist studies a number of specific experimental results and from these tries to reach a conclusion which will hold for all cases. By reasoning from a number of specific experimental results, a general statement is developed which will apply in all similar situations. This process is called inductive reasoning. It should be used with caution. There may be occasions when a few examples suggest a conclusion which is not true in general. Thus, if you go to New York and meet five people in succession with red hair, it is not safe to conclude that everyone in New York has red hair. Also, if you notice that $\frac{1\cancel{9}}{\cancel{9}5} = \frac{1}{5}$, $\frac{1\cancel{6}}{\cancel{6}4} = \frac{1}{4}$, $\frac{4\cancel{9}}{\cancel{9}8} = \frac{4}{8}$, and $\frac{2\cancel{6}}{\cancel{6}5} = \frac{2}{5}$ you will be in trouble if you assume that you may always cross off numerals in this way.

When a general rule has been suggested, the scientist tries to verify it by further experiment and, if possible, by deductive reasoning. In all these steps mathematics and mathematical reasoning are especially important.

Let us look at the results in our tables from this point of view. We are trying to determine whether there seems to be a general rule which describes all these relationships. If possible, we wish to express the rule in mathematical terms. If it is a general law, we should be able to use it to predict where to place one object of known weight to balance a second of known weight.

Exercises 14-3

1. (a) In Table I, do you notice any connection between the location of masses of equal weight on opposite sides of the fulcrum? What is it?

   (b) If the value of W is doubled, w and d remaining unchanged, how does the corresponding value of D change?

   (c) If the value of W is made half as much, how does the corresponding distance D from the fulcrum change?

   (d) Do the values for w, d and W, D appear to be related in any way? Can you state a general rule that seems to hold concerning w, d, W, and D? State the rule in words and then in the form of a mathematical equation in which you use the symbols w, W, d, and D.

   (e) Check your rule by applying it to some of the entries found by experiment in Tables II, III, IV.

2. Use the equation suggested in the preceding exercise to predict the missing entries in Table V.

TABLE V

w	25	35	12	45	21	15	23	11		14	10	100	100	100
d	3	4	5	3	5		4	8	7	5.5	50	50	500	5000
W	15		7		14	20	12		13	10				
D		7		9		4.5		10	8		5	5	5	5

3. Go back to your experiment and check the results in Table V to see if they actually produce a balance.

14-4. Graphical Interpretation.

In Chapter 13 you learned about graphs and their usefulness in presenting numerical information in a clear and condensed way. Scientists often make a graph of the observations obtained through experimentation for the help it can give them in summarizing and interpreting the data.

In the equation $wd = WD$ which you obtained for yourself in the preceding experiment, four quantities are involved. This equation can be interpreted in a number of ways depending on the way you followed through in the experiment. In the first part of the experiment, you chose fixed values for w and d and then found the values for W and D which produced a balance. From each experiment you got one pair of values which satisfied the relation $WD = 120$. A graph of $WD = 120$, pictures many pairs of values which produce a balance when $wd = 120$. The graph, then, supplies not only the information in Table I but other possible values for W and D.

The next step is to draw the graph of the relation connecting a value of W and the corresponding value of D in Table I.

If you need help in drawing the graph, the following suggestions should aid you:

Use graph paper and begin with two perpendicular lines called axes. The intersection of the axes is named point O.

Label the horizontal axis W and the vertical axis D.

If you use $\frac{1}{4}$ inch squared paper, a suitable scale is one for each space.

In Table I, the first value of W is 10 and the corresponding value of D is 12. Locate 10 on the W axis. Follow the vertical line through 12 on the D axis. This point is called (10, 12). Dotted lines in the graph in Figure 14-4a will help you locate this point. Make a small dot for the point.

Similarly, from Table I, when $W = 20$, $D = 6$. Locate 20 on the W axis. Follow the vertical line through 20 to the point where it meets the horizontal line through 6 on the D axis. This point is called (20, 6).

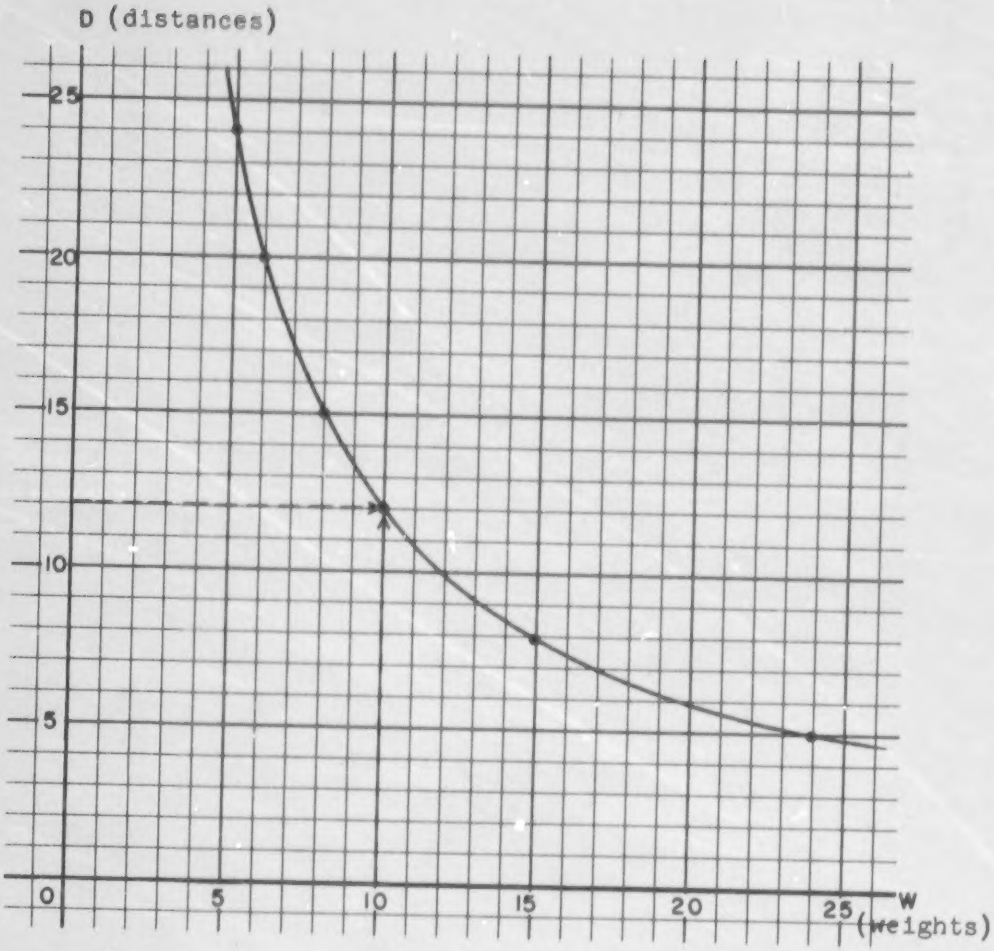
Before you read on, locate and mark the other points from Table I. These are (5, 24), (8, 15), (24, 5), (12, 10).

Mark the points which correspond to the results you found in columns (h), (i), and (j) in Table I.

Fill in the blanks in the following pairs for (W, D) and mark the corresponding points on your graph: (4,), (, 6), (16,), (, 18). Use $WD = 120$.

Draw a smooth freehand curve through the points you have located. This curve gives you the general picture of the relation between weights and distances as in Table I. If any point seems to lie to one side or the other of your smooth curve, check your computation. Not all experiments turn out perfectly and not all results fall into neat patterns at once. The points obtained from the measurements you made should fall near the curve. Often scientists expect no more than this from an experiment of this type.

The curve that you have drawn is a portion of a curve called a hyperbola. You will learn more about this curve in your study of algebra.



Graph of WD = 120
from Data in Table 1

Figure 14-4a

Exercises 14-4

1. Study the graph and then answer these questions:
   (a) If the value of w is increased, how does the corresponding value of d change?
   (b) If the value of d is increased, how does the corresponding value of w change?

2. In order to find the value of W when D is 24, locate 24 on the axis and follow the horizontal line through 24 until it meets the curve. Read the value on the W scale directly beneath this point. You should have 5. Find the values of W from the graph for the following points:
   (a) (, 20) (b) (, 15) (c) (, 6)

3. Find the values of D from the graph:
   (a) (6,) (b) (4,) (c) (15,)

4. Tell whether or not each of the following points is on the graph:
   (a) (10, 25)
   (b) (15, 8)
   (c) (5, 5)
   (d) (20, 15)

5. Estimate the missing values:
   (a) (7,)
   (b) (8,)
   (c) (, 9)
   (d) (17,)
   (e) (, 21)
   (f) (23,)

6. Draw a graph of the relation between W and D given in Table II. Use the formula $WD = 96$ to find the number pairs you need for locating points. Check the values you find with those in Table II.

7. From the graph find D when W is 20; find W when D is 12.

8. How does the value of d change when the value of w is decreased? Increased?

9. Does this graph have anything in common with the graph you drew for WD = 120?

14-5. Other Kinds of Levers.

In the introduction to this chapter you read that not all levers are like seesaws. You may be interested in asking about this in your science class at an appropriate time. In addition to the automobile jack, bottle opener, and crowbar mentioned before, there are other examples of widely used forms of the lever. You may think of ice tongs, nutcrackers, scissors, claw hammers, pliers, pruning shears, and hedge clippers as some useful levers.

14-6. The Role of Mathematics in Scientific Experiment.

Although the experiment using the lever does not use a great deal of mathematics, it does suggest how mathematics is used in scientific activities. You saw how mathematics was used in measuring, counting, and comparing quantities. You noted how observations of data were recorded in mathematical terms.

You searched for a pattern by studying the numbers in your recorded data. By reasoning from a set of specific cases you developed a general statement to be applied in all similar situations. In Section 14-3 this kind of reasoning is called inductive reasoning. It leads from a necessarily restricted number of cases to a prediction of a general relationship. This general relationship was stated in mathematical symbols in an equation: WD = wd. To establish this general principle, further experimentation was performed.

In addition, you drew a graph of WD = 120 and of WD = 96 to show how these statements tell the complete story in each case. The graph is another instance of the use of mathematics to interpret and to summarize a collection of facts.

The graph also helped to reveal the general pattern which was discovered.

Many scientific facts were undiscovered for thousands of years until alert scientists carefully set up experiments much as you have done and made discoveries on the basis of observations. Some examples of these are the following:

(a) Until the time of Galileo, people assumed that if a heavy object and a light object were dropped at the same time, the heavy one would fall much faster than the light one. Look up the story of Galileo and his experiment with falling objects and see what he discovered.

(b) From time immemorial, people watched eclipses of the sun and moon and saw the round shadow of the earth but did not discover that the earth was round. Eratosthenes, in 230 B.C., computed the distance around the world by his observations of the sun in two locations in Egypt, yet seventeen hundred years later when Columbus started on his journey, many people still believed the world was flat. Look up in a history of mathematics book or in an encyclopedia the story of Eratosthenes and this experiment.

(c) People had watched pendulums for many centuries before Galileo did some measuring and calculating and discovered the law which gives the relation between the length of the pendulum and the time of its swing. Look up this experiment in a book on the history of mathematics or of science.

Notice that all these experiments are based on many careful measurements and observations in order to discover the scientific law. Then the law is stated in mathematical terms. A great deal of science depends upon mathematics in just this way.

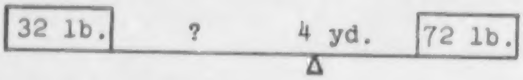
The examples which we have given here describe older fundamental discoveries all of which used relatively simple mathematics. The scientists of today are using more advanced mathematics, and many of the newer kinds of mathematics, in their scientific experiments.

Exercises 14-6

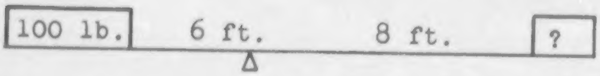
1. Some seesaws are fitted so that they can be shifted on the support. Why?

Find the missing value in each case:

2.



3.



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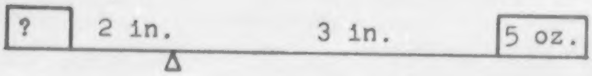
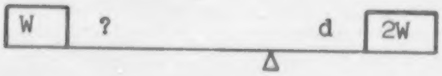


5.



6. Do you suppose a 90 lb. girl could ever lift a 1000 pound box? Justify your answer.

7. A child who weighed 54 pounds asked his father (180 pounds) to ride a seesaw with him. Where should the father sit to balance the child?

8. An iron bar 6 feet long is to be used as a lever to lift a stone weighing 75 pounds. The fulcrum is 2 feet from the stone. A force of how many pounds is needed to lift the stone?

9. Suppose a 100 pound boy sits on one end of a six foot crowbar, $5\frac{1}{2}$ feet from the fulcrum. Disregarding the weight of the bar, how heavy an object can he lift at the other end of the bar?

10. Which points lie on the graph of $WD = 60$?

(a) (12, 5) (b) $(\frac{1}{3}, 180)$ (c) (1, 6)

Bibliography

1. Newman, James R. THE WORLD OF MATHEMATICS. New York: Simon and Schuster, 1956.

   (Galileo, pp. 726-770; falling bodies, p. 729; pendulum, pp. 729, 741; Erathosthenes, pp. 205-207.)

2. Kline, Morris. MATHEMATICS IN WESTERN CULTURE. New York: Oxford University Press, 1953.

   Chapter XIII The Quantitative Approach to Nature
   Chapter XIV The Deduction of Universal Laws

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